Final Technical Report

ANALYTICAL AERIAL TRIANGULATION ERROR ANALYSIS AND APPLICATION OF COMPENSATING EQUATIONS TO THE GENERAL BLOCK TRIANGULATION AND ADJUSTMENT PROGRAM

MARCH 1960 TO FEBRUARY 1962
DEPARTMENT OF THE ARMY TASK NO. 8T35-11-001-05
CONTRACT NO. DA-44-009 ENG 4420

U.S. Army Engineer
Geodesy, Intelligence, and Mapping
Research and Development Agency

MIT

DEPARTMENT OF CIVIL ENGINEERING

SCHOOL OF ENGINEERING
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Cambridge 39, Massachusetts

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ANALYTICAL AERIAL TRIANGULATION ERROR ANALYSIS AND
APPLICATION OF COMPENSATING EQUATIONS TO
THE GENERAL BLOCK TRIANGULATION AND ADJUSTMENT PROGRAM

FINAL TECHNICAL REPORT

Civil Engineering Systems Laboratory
Department of Civil Engineering
Massachusetts Institute of Technology
Cambridge, Massachusetts

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Civil Engineering Systems Laboratory, Department of Civil Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts - ANALYTICAL AERIAL TRIANGULATION ERROR ANALYSIS AND APPLICATION OF COMPENSATING EQUATIONS TO THE GENERAL BLOCK TRIANGULATION AND ADJUSTMENT PROGRAM - L. Andrew R., Z.M. Elias, F.S. Greatorex

Final Technical Report, 1 August 1962, 145 pages, Corps of Engineers, GIMRADA, Contract No. DA-44-009 ENG 4420, Task No. 8T35-11-001-05, Unclassified Report

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1. the nature of random and systematic errors and the basic techniques for treating their effects as applicable to the analytical photogrammetric problem,
2. the basic least squares method and its incorporation in the computer program,
3. complete mathematical description of the program,
4. studies of the nature and effects of the important error sources: lens and camera errors, atmospheric refraction, film distortion,
5. the study of various techniques for the solution of simultaneous equations,
6. operating instructions,
7. the results, conclusions, and resulting recommendations of test runs of the final computer program.

The second volume contains the appendices which consist of the complete flow charts representing the original and final programs.

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1. Mapping
Photogrammetry,
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CIVIL ENGINEERING SYSTEMS LABORATORY

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PREFACE

These investigations were carried out under the authority contained in Contract No. DA-44-009 ENG 4420, "Analytical Aerial Triangulation Error Analysis and Application of Compensating Equations to the General Block Triangulation Adjustment Program."

The investigation was conducted under the direction of Professor Charles L. Miller and Mr. E. Phillip Gladding, Instructor, Department of Civil Engineering. The investigation was conducted, from March 10, 1960 to March 10, 1962, by Messrs. Luis Andrew R., Ziad M. Elias, and Frank S. Greatorex, Research Assistants in the Department of Civil Engineering. Also contributing to the investigation were Messrs. Daniel R. Schurz, Instructor, Armen Gabrielian, Student Assistant, and Lawrence Kalman, Student Assistant.

ABSTRACT

The objective of the activities reported is to effect improved accuracy in the supplied General Block Triangulation digital computer program through incorporation in the program means of error adjustment and compensation. The first volume of the report presents:

1. The nature of random and systematic errors and the basic techniques for treating their effects as applicable to the analytical photogrammetric problem;

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SUMMARY

This report describes research directed toward the general problem of adjustment and compensation of errors in the analytical aerial photogrammetric triangulation problem. The primary objective is the incorporation in the General Block Triangulation and Adjustment Program, supplied by the government, the necessary routines for effective adjustment of random errors. The report includes studies of the particular error sources in the photogrammetric system.

The general method of incorporating the least squares adjustment of random errors is presented along with a detailed mathematical description of the final program. The program has been tested using real and fictitious data. While the program is effective in adjusting for random errors, the systematic error content of the available data is dominant. Considerable improvement in the compensation of lens and film errors is required. A procedure is recommended for compensating for atmospheric refraction effects.

It has been shown that:

1. a general routine can be used to compensate for both vertical and non-vertical photography;

2. very high altitudes present no additional problems;

3. four fiducial are not sufficient for film distortion correction - more control is required;

4. the Jordan Diagonalization method for solving simultaneous equations reduces the operating time;

5. analytical or numerical representations of lens distortion may be used to implement radial lens distortion correction.

I. INTRODUCTION

A. Project Objectives

The primary objectives of this contract are to: (1) study and evaluate the errors associated with analytical photogrammetry for the purpose of, (2) incorporating a means of analytically correcting and/or compensating for those errors into the ERDL General Block Triangulation and Adjustment Program.

This report presents the results of the random and systematic error studies and the resulting programming incorporated into the subject computer program, with an evaluation of the effectiveness of the resultant program.

B. Sources and Treatment of Errors

1. Scope

A typical complaint associated with technical reports is that the author and reader assign different interpretations to fundamental terms and concepts. Such a situation exists in the consideration of errors. The following paragraphs of this section are intended to give the reader a brief review of basic concepts involved in the analysis and treatment of errors.

There are three characteristically different sources of variance between the actual value of a physical quantity and the value obtained through measurement. A brief discussion

of each of these three follows.

2. Blunders

Blunders are mistakes made during the measuring process or during subsequent handling of the data. Blunders are usually human mistakes. There is no general approach or mathematical method for detection, compensation, or elimination of blunders. Blunders must be detected by some independent checking method and eliminated before applying systematic and random error considerations. Blunder-checking techniques can only be determined by analysis of the involved measurement problem and system.

3. Systematic Errors

Systematic errors are those errors contained in measured quantities which are deterministically caused by the conditions of measurement; that is, the conditions at the time and place of the measurement which deterministically alter the value intended to represent a measured quantity and result in systematic errors. This is a relatively succinct and rigorous definition. If a certain condition affects the results of a measurement, the condition not being part of that intended to be measured, then the component resulting from the condition is an error. These conditions are commonly called error sources. The relationship between cause, the error source or condition, and the effect -- the error -- must be known. The effect must be a deterministic function of the condition. This implies that the value of the error

may be computed. This is true if the source condition can be measured and if the relationship between the condition and its effect is known. If, however, one of these requirements cannot be satisfied but the effect itself can be measured, then the effect of the source condition is determined with equivalent utility.

4. Random Errors

Random errors are those errors which obey the laws of chance. A common assumption is that random errors are Gaussianly distributed. The important general implications of this assumption are:

(1) Small errors are more likely than large errors.

(2) Positive and negative errors are equally likely.

Unlike systematic errors, random errors cannot be determined per se. Consideration of random errors is based upon the assumption that they are distributed Gaussianly. Rigorous considerations of random error propagation and the adjustment of related measured quantities which contain random errors are results of the Gaussian distribution expression. The resulting techniques and procedures are part of what is normally known as the Method of Least Squares.

5. Practical Error Conditions

Before any mathematical error analysis can be performed on measured data, all blunders must have been removed. The

measurement data is then assumed to be blunder-free. The next step is the removal of systematic errors. Each piece of measured data is analyzed and corrected for every applicable systematic error source. A condition for rigorous application of the random error considerations is that all systematic errors have been removed. One must, therefore, be able to determine the values of all systematic errors; but this is not always possible. Results of random error considerations are not always what they should be because of the unavoidable presence of systematic errors. Once all the systematic errors that can be determined have been removed, then random error adjustment may be applied, remembering that it is no better than the degree to which the assumptions have been met. The final result of the random error adjustment of the measured quantities is based, unlike systematic error correction, on the relationship between the measured quantities. The adjustment criteria, a result of the Gaussian expression, is that the sum of the squares of the weighted differences between the observed and adjusted values for all measured quantities is minimized. The weight associated with each measurement is inversely proportional to the square of the standard deviation for that particular measurement. This, again, is a result of the Gaussian assumption.

II. RANDOM ERROR ADJUSTMENT IN ANALYTICAL PHOTOGRAMMETRY

A. Mode of Attack

Accuracy in analytical photogrammetry is related to two considerations: first, the underlying geometrical representation of the problem; second, the accuracy of the input data. The geometrical representation assumes that light rays are straight lines and pass without deviation through a geometrical point. Atmospheric refraction and lens distortion introduce errors in this assumption.

Film shrinkage, instrumental as well as biased errors in measuring the coordinates of an image point and other information such as ground control point coordinates introduce errors in the input data. The determination of ground point coordinates from the locations of its two images on two photographs will be affected by two classes of errors. One is present in the elements of exterior orientation and depends on the factors discussed above. The other is introduced through the discrepancy between the measured photo image coordinates and the theoretically correct ones.

When they are systematic, the errors can be eliminated by a correction, but only to a certain degree. What is left are unknown random errors and residuals from systematic errors. The only way which is seen to study them and their effects is to study their probabilistic behavior and consider all of them as random errors. We will consider, therefore, that we have a set of random errors in the input data. In case of redundant data, the adjustment will be carried according to the method of least squares.

B. The Dodge Method

In this method, residuals* are not considered. For example, when the condition for rays O_1I_1 and O_2I_2 to intersect is expressed, the positions of images I_1 and I_2 in their respective photographs is assumed to be exactly known. More clearly, in the iterative process of solution only the exterior orientation of a bundle of rays is altered at each cycle, but the interior orientation is fixed. As the primary sources of errors are in the location of the theoretical image points, a rational adjustment by the method of least squares should take into account the residuals of the photo image coordinates.

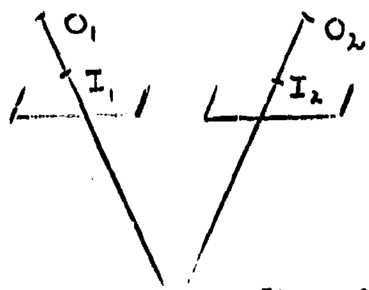

Figure 1a

In the following, these residuals and those of ground control data will be formed on the condition that the sum square of the weighted residuals is minimum.

C. Remarks on the Notation and the Derivation

A nondefined notation has the same meaning as in the Dodge report: "Sequential Presentation of Analytical Aerotriangulation by the Revised Direct Geodetic Restraint Method." In the following derivation of the completed equations, the same order of presentation

*The word "residuals" is used here and subsequently to indicate the measurement errors.

as in the above will be followed and the same equation numbers will be used. The reader is assumed to be familiar with the notation and the derivation of the equations of the Dodge report.

The residuals

These are of two kinds:

1. residuals in a photograph -- 2 per image dI_x and dI_y
2. residuals on the ground -- 1, 2, or 3 per control ground point: $d\phi$ and/or $d\lambda$ and/or dh

The number of residuals in a given equation depends on the number of images and the type of the ground point involved. See Chart 1.

D. Method of Incorporation of Least Squares Adjustment

The need to compensate for random errors initiated an examination of the least squares method. A set of equations is said to be redundant when the number of equations (m) exceeds the number of unknowns (n). In theory, there should be one, and only one, explicit solution. However, because of measuring errors in the measured data, one could arrange the redundant set into $S = \frac{m!}{n!\,(m-n)!}$ subsets of n equations with n unknowns, the result being S independent solution sets, with each solution vector being different. To establish a means of obtaining a single solution vector for the redundant set, the method of least squares was introduced. The classic approach is to assign to each equation a residual term V equal to the difference between the equation real and observed values; that is, $V = E - AX - BY - \ldots - FZ$, and then to minimize the weighted sum square of these residual terms. When

EQUATION		FIRST OR SECOND STRIP						SECOND STRIP				GROUND CONTROL		
		1st IMAGE		2nd IMAGE		3rd IMAGE		1st IMAGE		2nd IMAGE				
1	PASS POINT	dl_{1X}	dl_{1Y}	dl_{2X}	dl_{2Y}									
2	DIFF. SCALE RESTRAINT	dl_{1X}	dl_{1Y}	dl_{2X}	dl_{2Y}	dl_{3X}	dl_{3Y}							
3	COMPLETE GROUND CONTROL	dl_{1X}	dl_{1Y}									$d\phi$	$d\lambda$	dh
4		dl_{1X}	dl_{1Y}									$d\phi$	$d\lambda$	dh
5	KNOWN LONGITUDE	dl_{1X}	dl_{1Y}	dl_{2X}	dl_{2Y}								$d\lambda$	
6	KNOWN LATITUDE	dl_{1X}	dl_{1Y}	dl_{2X}	dl_{2Y}							$d\phi$	$d\lambda$	
6A	KNOWN LATI-TUDE ONLY	dl_{1X}	dl_{1Y}	dl_{2X}	dl_{2Y}							$d\phi$		
7	KNOWN ELEVATION	dl_{1X}	dl_{1Y}	dl_{2X}	dl_{2Y}									dh
56	KNOWN LAT.& LON. ONE IMAGE ONLY	dl_{1X}	dl_{1Y}									$d\phi$	$d\lambda$	
101	BLOCK ADJUSTMENT	dl_{1X}	dl_{1Y}	dl_{2X}	dl_{2Y}			dl^*_{1X}	dl^*_{1Y}	dl^*_{2X}	dl^*_{2Y}			
102		dl_{1X}	dl_{1Y}	dl_{2X}	dl_{2Y}			dl^*_{1X}	dl^*_{1Y}	dl^*_{2X}	dl^*_{2Y}			
103		dl_{1X}	dl_{1Y}	dl_{2X}	dl_{2Y}			dl^*_{1X}	dl^*_{1Y}	dl^*_{2X}	dl^*_{2Y}			

CHART 1

practiced, this then would give an adjusted solution vector which is said to be the most probable solution vector for the redundant set. The introduction of weights into this method is done by multiplying the equations by a set of predetermined numbers. In essence, this can be interpreted to mean that certain equations are stronger, i.e., have less measurement error than other equations within the redundant set. This can be proved to be equivalent to writing an equation a certain number of times more than a weaker equation. To summarize, this approach, then, gives greater significance to those equations known to have less measuring error, by minimizing the weighted sum square of the equation residual errors. This classical approach is the simplest case of least squares adjustment provided that an appropriate selection of weights for the redundant set of equations is possible. The question of weight selection proves to be the heart of the problem. The word weight is associated with a measured quantity and is a number inversely proportional to the square of the standard deviation or variance of this measured quantity.

To illustrate this, consider the following set of 3 equations with 2 unknowns, x and y:

$$a_{11}x + a_{12}y = b_1$$

$$a_{21}x + a_{22}y = b_2$$

$$a_{31}x + a_{32}y = b_3$$

Assume that the coefficients of the equations are known to be without error, but that the constant terms b_1, b_2, b_3 are determined through direct measurements having standard deviations of J_1, J_2, and J_3 units respectively. The classical least squares method would solve the

following set of equations:

$$a_{11}x + a_{12}y - V_1 = b_1$$

$$a_{21}x + a_{22}y - V_2 = b_2$$

$$a_{31}x + a_{32}y - V_3 = b_3$$

and the solution x_o, y_o satisfies the following condition:

of all the possible pairs of 2 numbers x_o, y_o the solution is that pair which when substituted in the following expressions

$$V_1 = a_{11}x_o + a_{12}y_o - b_1$$

$$V_2 = a_{21}x_o + a_{22}y_o - b_2$$

$$V_3 = a_{31}x_o + a_{32}y_o - b_3$$

$$\text{makes} \left(\frac{V_1^2}{J_1^2} + \frac{V_2^2}{J_2^2} + \frac{V_3^2}{J_3^2} \right) \text{a minimum}$$

In the case where both the coefficients and the constant terms are measured quantities or functions of measured quantities, the problem becomes more complicated.

This is the case in a problem such as the "General Triangulation and Adjustment Program" where errors are introduced into the coefficients and into the constant terms through the control data. Accordingly, a method which would use as criteria, (1) the minimization of the measurement errors themselves, and (2) would weight each measured value as a function of the expected magnitude of its measurement error, was desired. This method was found in a report (1) presented by D.C. Brown.

Before proceeding into the main of the report, it will be profitable

to review briefly this new least squares method.

Let m_o = the measured (observed) values

m = the true value of the measured items

v = the measuring errors

thus, $m = m_o + v$

Let x_o = the approximated parameter value

x = the true parameter value

r = the parameter residual

then, $x = x_o + r$

Accordingly, the conditional equations can be expressed as:

$$= f\ (x_o + r,\ m_o + v)$$

when the conditional equations are approximated by the first two terms of a Taylor expansion where:

$$\xi = f\ (x_o,\ m_o)$$

$$a = \frac{\delta\xi}{m}$$

$$b = \frac{\delta\xi}{\delta x}$$

the linearized equations take the form of

$$b_{11}r_1 + b_{12}r_2 + \ldots + a_{11}v_1 + a_{12}v_2 + \ldots + \xi_1 = 0$$

Now the primary approach is to minimize the weighted sum of the squares of the measurement errors or

$$s = w_1 v_1^{\ 2} + w_2 v_2^{\ 2} + \ldots$$

where $wj = \frac{\sigma o^2}{\sigma j^2}$

σo^2 = the unit variance

and σj^2 = the observation variance.

For convenience with matrix notation, let σ^o be the diagonal matrix whose elements are the inverse of the w_j

A = the matrix of the "a" elements

B = the matrix of the "b" elements

v = the column matrix of the measurement errors

r = the column matrix of the unknown parameter residuals

$\mathcal{E}$ = the column matrix of the linearized conditional equation values.

Then equation (7) can be written as

$$Br + Av + \mathcal{E} = 0$$

and

$$S = V^t (\sigma^o)^{-1} V$$

At this stage, Lagrange multiplier's are introduced to aid in the obtaining of a solution; thus, let λ = the row matrix of Lagrange multipliers and rewrite (11) as:

$$S = V^t (\sigma^o)^{-1} V - 2 \lambda (Br + Av + \mathcal{E})$$

The conditions that S be minimum are obtained by equating to zero the partial derivatives of S with respect to the measurement errors V and the parameters residuals r.

This leads to:

$$(\sigma^o)^{-1} V - A^t \lambda = 0$$

and $B^t \lambda = 0$

solving (13) for V gives

$$V = \sigma^o A^t \lambda.$$

and (15) substituted into (10) gives

$$Br + (A \sigma^o At) \lambda + \mathcal{E} = 0$$

Then, solving (16) for λ gives

$$\lambda = - (A \sigma^o At)^{-1} (Br + \mathcal{E})$$

and (17), substituted into (14) gives finally

$$(B^t(A\sigma^o A^t)^{-1}B)\ r + B^t\ (A\sigma^o A^t)^{-1}\ \mathcal{E} = 0$$

To check the matrix dimension validity, let

P = the number of unknown parameters

EQ = the number of redundant conditional equations

Q = the number of measurement errors

Then, the individual matrix dimensions are:

$B = (EQ \times P)$

$A = (EQ \times Q)$

$\sigma^o = (Q \times Q)$

$r = (P \times 1)$

$\mathcal{E} = (EQ \times 1)$

Thus,

$$(A\sigma^o A^t)^{-1} = (EQ \times Q)\ (Q \times Q)\ (Q \times EQ) = (EQ \times EQ)$$

$$B^t\ (A\sigma^o A^t)^{-1} = (\ P \times EQ)\ (EQ \times Q) = (P \times EQ)$$

Therefore,

$$(B^t\ (A\sigma^o A^t)^{-1}\ B)\ r = (P \times EQ)\ (EQ \times P)\ (P \times 1) = (\ P \times 1)$$

and

$$B^t\ (A\sigma^o A^t)\ \mathcal{E} = (P \times EQ)\ (EQ \times 1) = (P \times 1)$$

Thus, the above method allows one to minimize the weighted sum square of the physical errors associated with the measured (observed) data without increasing the number of unknowns over that defined by the classic least squares method. Nor does it, in any way, alter the ERDL selected approach for solving the set of normal equations.

The basic difference in computer mathematical logic is in the superposition of the redundant equations in forming the normal equations. With the classic least squares method, each individual linearized

equation could be normalized and then its contribution superimposed additively into the normal equations. However, with this new approach, all linearized equations associated with a particular point must be normalized as a set. Once this is done, the normalized set's contribution can be superimposed additively into the normal equations.

In the "General Block Triangulation and Adjustment Program", the coefficients of the equations derived by H. F. Dodge represent the elements of the aforementioned "B" matrix. Thus, the work required of this laboratory was to derive additions to these equations which then would supply the elements of the "A" matrix, i.e., the matrix of the coefficients of the unknown measurement errors.

III. LENS AND CAMERA ERRORS

A. Focal Length

The computer program, by its mathematical organization, requires the use of a constant focal length value. Therefore, variable focal length techniques to compensate for symmetric radial lens distortion cannot be employed here. The symmetric radial distortion curves presently being supplied by camera calibrations are plots of distortion vs. radial distance. This requires the use of a constant arbitrary focal length to construct the curve. By definition, this focal length is called the calibrated focal length. In regards to a small focal length change, it has been demonstrated that the resulting propagated error is very small.

B. Asymmetric and Tangential Lens Distortions

While it has become apparent that asymmetric and tangential distortions can be simulated and be made to behave as systematic errors, their distortion pattern orientations on the photograph could be a matter of chance (such as the prism effect from removable filters). In actual practice, there appear to be too many variables within the system to allow for an economical analytic treatment of these error sources for each job. Accordingly, it would seem that the proper means of avoiding these sources (following past operational procedures) lies in the "acceptance tolerances" defined for the optics and camera manufacturers.

C. Principal Point

Three different determinations of principal points are of interest.

They are: (1) indicated principal point, (2) autocollimated principal point, and (3) point of symmetry.

The indicated principal point is defined as that point formed at the intersection of two lines joining opposite fiducial marks.

The autocollimated principal point is that point on the photo plane at which the zeroth ray (during calibration testing) is imaged.

The point of symmetry is that statistically determined point from which the symmetric radial distortion curve is origined.

Examination of a typical planigon lens calibration test shows the point of symmetry to be +15 microns in x and -9 microns in y from the indicated principal point, and the autocollimated principal point to be +8 microns in x and -6 microns in y from the indicated principal point.

While it is not clear that the discrepancies in principal point location can be neglected, it is worthwhile to point out that it has been demonstrated that a small principal point displacement (5 microns in x and 8 microns in y) has a negligible effect on the final results (ground point coordinates).

Since the lens calibration curve is referenced from the point of symmetry, the calculations of image corrections for radial symmetric lens distortion will also have to be referenced from this point. The images themselves will, as is normal, be measured relative to the indication principal point. These coordinates should then

be translated to be referenced from the autocollimated principal point as this point is, optically speaking, the image of the ray passing through the perspective center and perpendicular to the film plane.

D. Radial Symmetric Lens Distortion

1. Investigation

The image coordinate shift due to this systematic error can be directly correlated to the distortion curves supplied by the camera calibrator. This curve is a plot of distortion versus the readial distance of the image from the "point of symmetry." The main point of concern, analytically speaking, is how to best represent this curve internally within the computer program.

The most obvious approach is to determine the coefficients of an n order polynomial which will mathematically simulate the graphic curve.

If d = lens distortion

X_s = the distance in the x direction of the image from the "point of symmetry"

Y_s = the distance in the y direction of the image from the "point of symmetry"

then r, the radial distance from the "point of symmetry is

$$r = (X_s^2 + Y_s^2)^{1/2}$$

and the equation for the distortion could be of the form:

$$d = Ar + Br + Cr^2 + Dr^3 + Er^4 + Fr^5$$

The computation of the coefficients (A......F) would be input parameters to the program, and they would have to be determined only after each camera calibration test.

The primary objection to the above treatment is that, for certain classes of lenses (such as for the aviator), the lens curve is not smooth nor simple, and the use of a single polynomial equation could only roughly approximate the curve.

While the present lens being employed in the photogrammetric system (the Planigon) has a smooth and simple curve, it is felt that the correction method selected should be general enough to treat the non-smooth, non-simple errors associated with lenses which might be introduced into the system in the future. A good working solution to treat this problem can be found in interpolation methods.

This laboratory has selected Newton's interpolation formula with divided difference, making use of four known points on the curve (for each interpolation) and using a third degree polynomial to calculate the distortion for points r microns from the "point of symmetry."

The procedure for this method is to first select and input into the computer the radial distance and related distortion for those points on the calibration curve which are necessary to define the curve (i.e., maximums, minimums, points of inflection, etc.). Then, when the distortion for a photograph image is desired, the computer program selects those four points (nominally two curve control points on either side of the image) which will best define that segment of the calibration curve pertinent to the image in

question. A third degree curve is fitted to these four points and the image distortion interpolated from this curve.

Let d and r = the distortion and radial distance of the image in question

d_o, r_o

d_1, r_1 = the distortion and radial distance of the curve control points selected by the program

d_2, r_2

d_2, r_3

then $d = f\ (r)$

$$= f(r_o) + (r - r_o)\ f\ (r_o, r_1) + (r - r_1)\ f\ (r_o, r_1, r_2) + (r - r_o)\ (r - r_1)\ (r - r_2)\ f\ (r_o, r_1, r_2, r_3)$$

where $f(r_o) = d_o;\ f(r_1) = d_1;\ f(r_2) = d_2;\ f(r_3) = d_3$

The divided difference interrelationship is:

$$\begin{tikzcd}[column sep=small]
r_o & f(r_o) \arrow[dr] & & & \\
& & f(r_o, r_1) \arrow[dr] & & \\
r_1 & f(r_1) \arrow[ur] \arrow[dr] & & f(r_o, r_1, r_2) \arrow[dr] & \\
& & f(r_1, r_2) \arrow[ur] \arrow[dr] & & f(r_o, r_1, r_2, r_3) \\
r_2 & f(r_2) \arrow[ur] \arrow[dr] & & f(r_1, r_2, r_3) \arrow[ur] & \\
& & f(r_2, r_3) \arrow[ur] & & \\
r_3 & f(r_3) \arrow[ur] & & &
\end{tikzcd}$$

thus giving that;

$$f(r_o,r_1) = \frac{f(r_1) - f(r_o)}{r_1 - r_o} = \frac{d_1 - d_o}{r_1 - r_o} = A_1$$

$$f(r_1,r_2) = \frac{f(r_2) - f(r_1)}{r_2 - r_1} = \frac{d_2 - d_1}{r_2 - r_1} = B_1$$

$$f(r_2,r_3) = \frac{f(r_3) - f(r_2)}{r_3 - r_2} = \frac{d_3 - d_2}{r_3 - r_2} = C_1$$

$$f(r_o,r_1,r_2) = \frac{f(r_1,r_2) - f(r_o,r_1)}{r_2 - r_o} = \frac{B_1 - A_1}{r_2 - r_o} = A_2$$

$$f(r_1,r_2,r_3) = \frac{f(r_3,r_2) - f(r_1,r_2)}{r_3 - r_1} = \frac{C_1 - B_1}{r_3 - r_1} = B_2$$

$$f(r_o,r_1,r_2,r_3) = \frac{f(r_1,r_2,r_3) - f(r_o,r_1,r_2)}{r_3 - r_o} = \frac{B_2 - A_2}{r_3 - r_o} = A_3$$

Therefore:

$$\begin{aligned} d &= f(r) \\ &= d_o + (r-r_o) + (r-r_o)(r-r_1)A_2 + (r-r_o)(r-r_1)(r-r_2)A_3 \end{aligned}$$

Since the distortion is radial and it is in the same radial direction as the image, one can use similar triangles to recover the ΔX and ΔY components of the distortion.

Thus, $\dfrac{\Delta X}{X_s} = \dfrac{d}{r}$

or $\quad \Delta X = X_s(d/r)$

and $\quad \Delta Y = Y_s(d/r)$

Now, by definition, plus (+) distortion is radially outwards and minus (-) distortion is radially inwards relative to the "point of symmetry." Thus, to find the correct image coordinates, one must adjust the distorted measured coordinates inwards for plus (+) distortion and outwards for a minus (-) distortion. Accordingly, let:

X_c = corrected X coordinate

Y_c = corrected Y coordinate

Thus, $X_c = \pm(X_s - \Delta X)$ if X_s is (+), then X_c is (+)

$Y_c = \pm$ if X_s is (-), then X_c is (-)

if Y_s is (+), then Y_c is (+)

if Y_s is (-), then Y_c is (-)

2. Discussion

The use of the Newton method gives a general approach to this problem. It allows one to select the curve control points which best define the curve, irregardless of the Δr increment between the point, and provides a means of operating with lens curves which cannot be defined explicitly by a polynomial equation.

However, if the lens curve is a simple curve and can be represented by a polynomial, it is obvious that the use of the Newton method, instead of the polynomial approach, would require more computation time and thus would be lacking in systems efficiency.

3. Conclusions

1. If the lens distortion curve is a simple, a smooth curve, it is more efficient to use an explicit polynomial correction equation technique.

2. If the lens distortion curve is a non-smooth and/or non-simple, or if a general (all lens) correction technique is desired, then the "Newton Interpolation" technique is a logical method to be used.

IV. ATMOSPHERIC REFRACTION COMPENSATION

A. Introduction

Because of the spacial variation of the refractive index of the atmosphere, a light ray travels along a curved path. The central projection of a photograph does not, therefore, represent correctly the original bundle of light rays which traveled to the camera perspective center from the ground points whose images appear on the photograph.

B. The Problem

The problem of atmospheric refraction correction is that of determining the displacement of an image point on a photograph so that the straight line joining the perspective center and the ground point passes through the displaced image point.

The usual assumption that the refractive index is constant on concentric spheres will be made. The center of these spheres can be taken as the center of a spherical earth or as the center of curvature of the ellipsoid representing the geoid at the photographed area.

In the following only the significant results will be presented. The derivations of the formulae to follow and an otherwise more detailed treatment can be found in the Phase II Interim Technical Report of this contract.

C. Apparent Ground Displacement

In Figure 1:

A is a ground point

H is the exposure station

B is the nadir

A' is the point of intersection of the tangent to the light ray at H with the sphere (S) whose refractive index is equal to that of A

C is the center of S

The sphere (S) will be taken as a datum for altitudes.

Let

H be the altitude BH of the perspective center H;

h be the altitude of a point on the light ray;

n be the refractive index at altitude h;

n_H be the refractive index at the exposure station;

V_H be the angle HA', HB;

$\Delta\theta$ be the angle CA', CA;

i be the angle CA', A'H;

$\alpha = \cos\phi_0$;

$R =$ radius of the sphere (S)

$X_H = \frac{H}{R}$

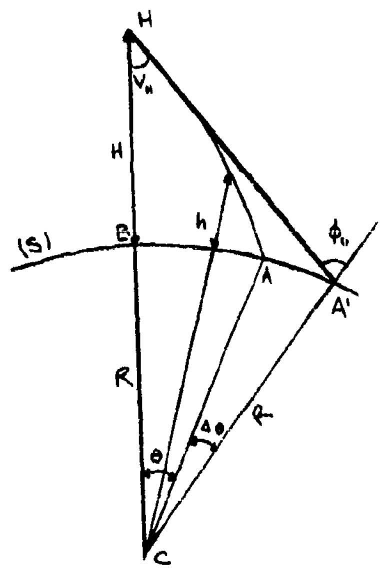

FIG. 1

The apparent ground displacement is arc AA' and is given by

$$AA' = L_1 - L_2 \tag{1}$$

where

$$L_1 = \frac{H\sqrt{1-\alpha^2}}{\alpha^3}\left(C_1 - \frac{3-\alpha^2}{\alpha^2} x_H C_2 + \frac{3}{2}\,\frac{5-3\alpha^2}{\alpha^4} X_H C_3 - \frac{1}{2}\,\frac{35-30\alpha^2+3\alpha^4}{\alpha^6} x_H{}^3 C_4\right) \tag{2}$$

$$L_2 = \frac{3H}{2}\,\frac{(1-\alpha^2)}{\alpha^5}^{3/2}\left(C_1' - \frac{5-\alpha^2}{\alpha^2} X_H C_2' + \frac{5}{2}\,\frac{7-3\alpha^2}{\alpha^4} X_H^2 C_3'\right) \tag{3}$$

The corresponding angular displacement $\Delta\theta$ (See Figure 1) is then

$$\Delta\theta = \frac{L_1 - L_2}{R}$$

The quantities $C_1, C_2, \ldots, C_1', C_2', \ldots$ appearing in (2) and (3) are defined as follows:

let

$$\frac{N_H}{N} = 1 - u \tag{4}$$

u is then a function of h. C_k and C_k' for $k = 1,2,3\ldots$ are defined by:

$$C_K = C_K(H) = \frac{1}{H^K}\int_0^H u h^{K-1}\, dh \tag{5}$$

$$C_K' = C_K'(H)\, \frac{1}{H^K}\int_0^H u^2 h^{K-1}\, dh \tag{6}$$

It should be noted that equations (1), (2), and (3) are the result of truncating infinite convergent series. Details on this question and the associated truncation error are to be found in [1].

D. Correction of Image Coordinates

The photographic coordinates x, y, z of an image point I are referenced to a rectangular right handed system of axes having the perspective center H as the origin. The photograph is here assumed to lie between the perspective center and the ground. In Figure 3, line HP is perpendicular to the photograph, P is the principal point and distance HP is the focal length f. Note that the z coordinate of any image point is negative and equal to $-f$.

Let Q be the intersection of the vertical (normal to the sphere (s)) passing through H and the plane of the photograph, and let I be the image point of a ground point A. As was mentioned earlier, the (curved) light ray HIA is in a plane containing HQ. The straight line HA intersects the photographic plane at point I_c which is the corrected image point. See Figures 2 and 3. The correction of the image point I consists of determining the components Δx and Δy of the vector $\overrightarrow{I_c I}$ so that denoting by x_c and y_c the coordinates of I_c will have:

$$x_c = x - \Delta x$$

$$y_c = y - \Delta y$$

Let n_1, n_2, n_3 be the direction cosines with respect to the photographic axes x, y, z of the vertical QH directed upwards. Let r = distance HI $= \sqrt{x^2+y^2+z^2}$ and let

$$S = \frac{2\cos\left(\phi_0 - \frac{\Delta\phi}{2}\right)\sin\frac{\Delta\phi}{2}}{\sin(\phi_0 - \nu_H - \Delta\phi)} \qquad (7)$$

Δx and Δy are given by

$$\Delta x = \frac{(n_1 z - n_3 x) r s}{z - n_3 r s} \qquad (8)$$

$$\Delta Y = \frac{(N_2 Z - N_3 Y)\, rs}{Z - N_3\, rs} \qquad (9)$$

Case of a vertical photograph

The case of a vertical photograph is obtained in letting

$n_1 = n_2 = 0$ and $n_3 = n_1$.

Then

$$\Delta X = -X \; \frac{rs}{Z - rs}$$

$$\Delta Y = -Y \; \frac{rs}{Z - rs}$$

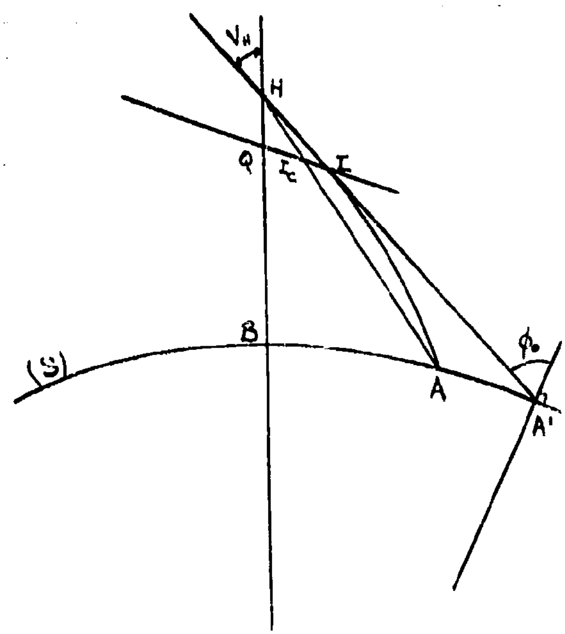

FIG. 2

and the displacement of the image -point I takes place on the line joining I to the principal point P and toward P (See Figure 4). The displacement II_c is then given by

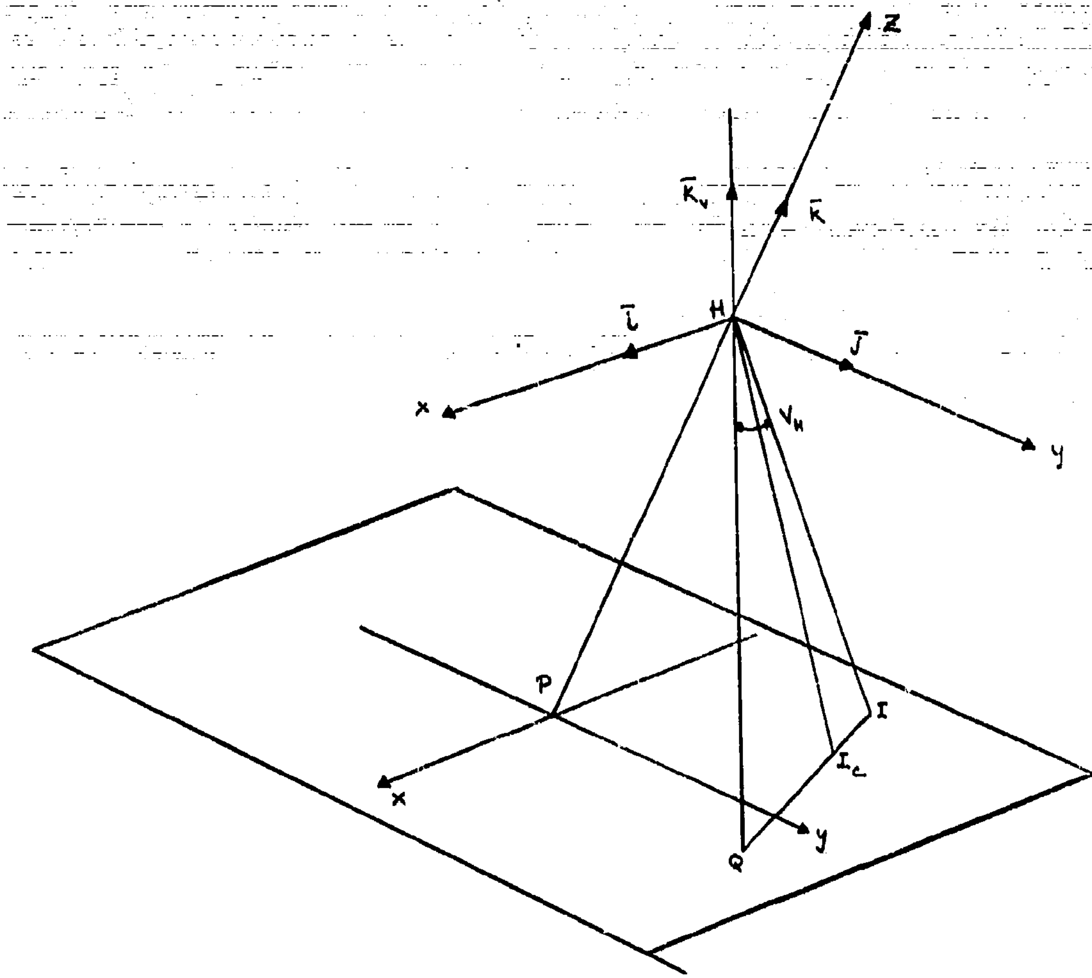

Fig. 3

$$II_c = f \frac{2\cos\left(\phi_0 - \frac{\Delta\theta}{2}\right)\sin\frac{\Delta\theta}{2}}{\cos V_H \left[X_H + 2\sin^2\frac{\theta_0 - V_H - \Delta\theta}{2}\right]} \qquad (10)$$

and

$$\Delta x = \frac{x}{\sqrt{x^2+y^2}} \cdot II_c \qquad (11)$$

$$\Delta y = \frac{y}{\sqrt{x^2+y^2}} \cdot II_c \qquad (12)$$

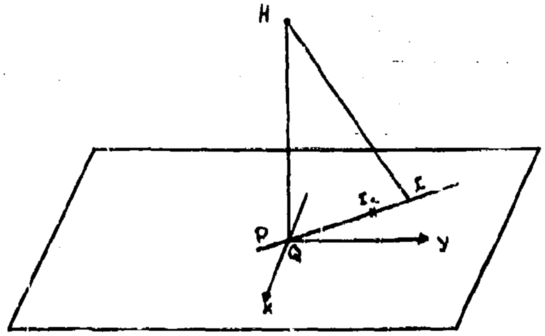

FIG. 4

Approximate formulae giving II_c in the case of a vertical photograph can be found in [7] where their accuracy is discussed.

E. Determination of $C_1, C_2, \ldots, C_1', C_2', \ldots$

These quantities are defined by:

$$C_K = \frac{1}{H^K}\int_0^H u h^{K-1}\, dh \qquad (5)$$

$$C_K' = \frac{1}{H^K}\int_0^H u^2 h^{K-1}\, dh \qquad (6)$$

where $k = 1,2,\ldots$

and $u = 1 - \frac{n_H}{n}$

let $M = n-1$

then

$$u = \frac{M-M_H}{1+M} = (M-M_H)\,(1-M+M^2-\ldots)$$

where the subscript H refers to the exposure station. We now make the approximation

$$u = M-M_H.$$

The corresponding error is discussed in [1]. M is related to the specific mass d of the atmosphere by the relation

$$\frac{M}{d} = \text{constant}.$$

Two atmospheres will be considered.

1. Idealized Atmosphere

The idealized atmosphere will be such that the specific mass d varies exponentially with altitude according to the equation:

$$d = d_o e^{-ah}$$

where

d_o = specific mass at the datum

a = suitable constant.

M will then satisfy the relation

$$M = M_o e^{-ah}$$

where

M_o = value of M at the datum.

Integration of (5) gives

$$C_1(H) \approx \frac{M_o}{aH}\left[1 - e^{aH}(1+aH)\right] \qquad (15)$$

$$C_2(H) = \frac{M_0}{(aH)^2}\left[1 - e^{-aH}\left(1 + aH + \frac{(aH)^2}{2}\right)\right] \quad (16)$$

$$C_3(H) = \frac{2M_0}{(aH)^3}\left[1 - e^{-aH}\left(1 + aH + \frac{(aH)^2}{2} + \frac{(aH)^3}{6}\right)\right] \quad (17)$$

$$C_4(H) = \frac{6M_0}{(aH)^4}\left[1 - e^{-aH}\left(1 + aH + \frac{(aH)^2}{2} + \frac{(aH)^3}{6} + \frac{(aH)^4}{24}\right)\right] \quad (18)$$

It may be interesting to note that C_{k+1} and C_k are related by the relation

$$C_{k+1} = \frac{k}{aH} C_k - \frac{M_H}{k+1}$$

Integration of (6) gives $C_k'(H)$ in terms of $C_k(2H)$ and $C_k(H)$.

$$C_k'(H) = M_0 C_k(2H) - 2M_H C_k(H) \quad (19)$$

where $M_H = M_0 e^{-aH}$.

If a is determined on the condition that d agrees with the ARDC Model Atmosphere at sea level h=0 and an altitude of 10 km., it is found that

$a = 0.10860\ \text{km}^{-1}$.

And at sea level for dry air, ARDC Model Atmosphere pressure and temperature, and for the Kodak filter 25(A), it is found that

$M_0 = 276.73 \times 10^{-6}$

In the preceeding, the altitude was measured with regard to a spherical datum on which the ground point lies. Thus, in the formulae giving $C_1(H)$, $C_2(H)$, ... ,$C_1'(H)$, $C_2'(H)$...

H is the altitude of the exposure station with respect to

the ground point and M_o is the value of M at the ground point. Thus when the exposure station and ground point altitudes with respect to sea level, say H_e and H_o, are given, we would have:

$$H = H_e - H_o$$

$$M_o = 276.73 \times 10^{-6} \times e^{-aH_o}$$

$$M_H = M_o e^{-aH} = 276.73 \times 10^{-6}\, e^{-aH_e}$$

2. ARDC Model Atmosphere

The density profile of the ARDC Model Atmosphere is given in [3] . Since $\frac{M}{d}$ = constant, we have

$$M = M_o \frac{d}{d_o} .$$

$\frac{d}{d_o}$ is tabulated in [3] and numerical integrations to determine C_1, C_2, ... , C_1', C_2', ... are possible. However, a direct tabulation of these quantities is not convenient since they depend not only on H, but also on the altitude of the ground point with respect to sea level. In the following, we will define quantities as functions of H only, from which C_1, C_2, ..., C_1', ... can be determined easily. Now (5) and (6) can be re-written respectively:

$$H^k C_k(H) = \int_0^H (M - M_H) h^{k-1}\, dh = \int_0^H M h^{k-1}\, dh - M_H \frac{H^k}{k}$$

$$H^k C_k'(H) = \int_0^H (M - M_H)^2 h^{k-1}\, dh$$

$$= \int_0^H M^2 h^{k-1}\, dh - 2M_H \int_0^H M h^{k-1}\, dh + M_H^2 \frac{H^k}{k}$$

If sea level is taken as datum and H_o and H_e are the altitudes of the ground point and exposure station respectively with respect to sea level, then we will have

$H^k = (H_e - H_o)^k$

and

$$(H_e - H_o)^k C_k = \int_{H_o}^{H_e} M h^{k-1}\, dh - M_H \frac{(H_e - H_o)^k}{k}$$

$$(H_e - H_o)^k C'_k = \int_{H_o}^{H_e} M^2 h^{k-1}\, dh - 2M_H \int_{H_o}^{H_e} M h^{k-1}\, dh + M_H^2 \frac{(H_e - H_o)^k}{k},$$

where the variable of integration h now has sea level as its origin. With this same convention, let

$$F_k(H) = \int_0^H M h^{k-1}\, dh$$

$$F'_k(H) = \int_0^H M^2 h^{k-1}\, dh$$

where values of F_1, F_2, F_3 and F'_1 for $H \leqslant 70$ km can be found in Table 1.

C_k and C'_k are then given by

$$(H_e - H_o)^k C_k = F_k(H_e) - F_k(H_o) - M_o \frac{d_e}{d_o} \frac{(H_e - H_o)^k}{k} \tag{20}$$

$$(H_e - H_o)^k C'_k = F'_k(H_e) - F'_k(H_o) - 2M_o \frac{d_e}{d_o} \left[F_k(H_e) - F_k(H_o) \right] + M_o^2 \frac{d_e^2}{d_o^2} \frac{(H_e - H_o)^k}{k} \tag{21}$$

Altitude : Km.	F_1	F_2	F_3	F_1'
2	5.0287108E-04	4.8642143E-04	6.3800878E-04	1.2684593E-07
4	9.1422930E-04	1.7063991E-03	4.3931411E-03	2.1175211E-07
6	1.2474444E-03	3.3604846E-03	1.2715018E-02	2.6748418E-07
8	1.5144552E-03	5.2194466E-03	2.5746245E-02	3.0328525E-07
10	1.7258620E-03	7.1136549E-03	4.2788760E-02	3.2573905E-07
12	1.8903649E-03	8.9155390E-03	6.2579977E-02	3.3935749E-07
14	2.0113081E-03	1.0481470E-02	8.2895441E-02	3.4673150E-07
16	2.0996597E-03	1.1802122E-02	.10266561	3.5066669E-07
18	2.1642152E-03	1.2896191E-02	.12122911	3.5276755E-07
20	2.2113929E-03	1.3790103E-02	.13818247	3.5388957E-07
22	2.2458778E-03	1.4512486E-02	.15332629	3.5448905E-07
24	2.2710894E-03	1.5091037E-02	.16661118	3.5480945E-07
26	2.2894953E-03	1.5550195E-02	.17807154	3.5498031E-07
28	2.3026975E-03	1.5905927E-02	.18766106	3.5506825E-07
30	2.3122143E-03	1.6181406E-02	.19563830	3.5511391E-07
32	2.3191339E-03	1.6395557E-02	.20226807	3.5513803E-07
34	2.3242064E-03	1.6562696E-02	.20777684	3.5515098E-07
36	2.3279540E-03	1.6693681E-02	.21235608	3.5515804E-07
38	2.3307433E-03	1.6796757E-02	.21616583	3.5516195E-07
40	2.3328341E-03	1.6878204E-02	.21933911	3.5516414E-07
42	2.3344119E-03	1.6942824E-02	.22198608	3.5516538E-07
44	2.3356102E-03	1.6994301E-02	.22419770	3.5516609E-07
46	2.3365260E-03	1.7035472E-02	.22604888	3.5516650E-07
48	2.3372306E-03	1.7068560E-02	.22760291	3.5516674E-07
50	2.3377828E-03	1.7095602E-02	.22892718	3.5516688E-07
52	2.3382180E-03	1.7117783E-02	.23005800	3.5516696E-07
54	2.3385614E-03	1.7135984E-02	.23102188	3.5516700E-07
56	2.3388370E-03	1.7151137E-02	.23185491	3.5516702E-07
58	2.3390586E-03	1.7163767E-02	.23257446	3.5516703E-07
60	2.3392357E-03	1.7174215E-02	.23319065	3.5516704E-07
62	2.3393763E-03	1.7182792E-02	.23371370	3.5516704E-07
64	2.3394872E-03	1.7189778E-02	.23415367	3.5516704E-07
66	2.3395739E-03	1.7195421E-02	.23452032	3.5516704E-07
68	2.3396412E-03	1.7199939E-02	.23482292	3.5516704E-07
70	2.3396930E-03	1.7203523E-02	.23507014	3.5516704E-07

TABLE 1

where M_o and d_o are the values of M and d at any given point, and d_e is the value of d at the exposure station. $\frac{d}{d_o}$ is given in Table 2 with d_o referring to sea level. Then

$M_o = 276.73 \times 10^{-6}$

for the Kodak filter 25(A), dry air, and ARDC pressure and temperature at sea level. If a different value of M_o, say M_o', but still the same altitude should be used, F_k should be corrected by the factor $\frac{M_o'}{M_o}$ and F_k' by the factor $\left(\frac{M_o'}{M_o}\right)^2$

F. Application of Equations

The following data should be known or estimated in order to apply the atmospheric refraction correction.

1. Altitude of the exposure station with regard to sea level H_e.
2. Altitude of the ground point with regard to sea level H_o.
3. The direction cosines n_1, n_2, n_3 of the vertical passing through the exposure station with regard to the photographic axes.
4. Photographic coordinates of the image point $x, y, z = -f$.
5. Local radius of the earth R_o.

Knowing H_o and H_e, the constants C_1, C_2, ..., C_1', ... are determined for either the idealized or the ARDC atmosphere using equations (15) through (19) for the former and Table 1 with equations (20) and (21) for the latter. V_H is determined by

$$\cos V_H = - \frac{1}{\sqrt{x^2 + y^2 + z^2}} (n_1 x + n_2 y + n_3 z)$$

The datum sphere passes through the ground point. Its radius is then R_o+H_o and $x_H = \frac{H}{R} = \frac{H_e-H_o}{R_o+H_o}$.

ϕ_o is determined by

$(1+x_H)\sin V_H = \sin\phi_o$

then with

$H = H_e-H_o$

$\alpha = \cos\phi_o$

Formulae (2) and (3) are used to compute

$$\Delta\theta = \frac{L_1-L_2}{R} = \frac{L_1-L_2}{R_o+H_o}.$$

s is then computed using formula (7), use of (8) & (9) produces the corrections Δx and Δy, and finally

$x_c = x-\Delta x$

$y_c = y-\Delta y$

Table I contains F_1, F_2, F_3 and F_1' only. For all practical purposes, the terms in C_4,... and in C_2', C_3',... can be deleted. An upper bound of the corresponding truncation error is given in [7], p. 77.

Beyond a certain altitude H_L, the refractive index becomes close enough to unity so that refraction can be neglected in the region above H_L. In Figure 5 the light ray is HH_LA. H_LH can be considered as a straight line. It is seen that for $H > H_L$, the displacement $\overline{AA'}$ on the ground and the corresponding angular displacement $\Delta\theta$ are independent of the altitude H. Thus $AA' = L_1-L_2$ or $\Delta\theta = \frac{L_1-L_2}{R}$ are calculated using equations (2) and (3) as if

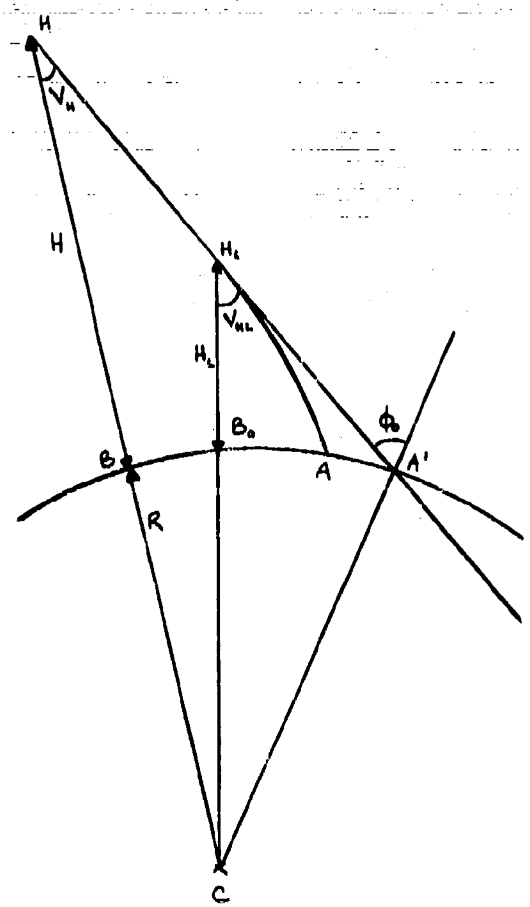

Fig. 5

the exposure station were at H_L. The angle V_{HL} is obtained from the angle V_H by the relation

$$(R+H_L)\sin V_{H_L} = (R+H)\sin V_H.$$

But when using (7) to compute s, V_H should be used.

G. Incorporation of a Compensation Routine in the Object Program

Some of the information needed to make the corrections, such as the altitudes of the camera station and of the ground point and the orientation of the vertical with regard to the photographic axes can only be estimated at the beginning of a job. The solution to this difficulty is to include the routine for atmospheric refractions in the iteration cycles of the program whereby the altitudes and orientations obtained from the resection at the end of a cycle are used to compute the corrections to every image point coordinates before these are processed in the next cycle. However, the inclusion of the iteration process should not be executed before a sufficient convergence has been achieved.

In case of near vertical photography and a reasonable estimate of the altitudes, the correction can be made only once, before the image point coordinates are processed. In that case, a simpler formula can be used to compute the image point displacement provided some conditions on the magnitude of H and V_H are met. See [7]. It should not be overlooked that the atmospheric refraction corrections should be made preferably along with other systematic error corrections, such as lens distortion and film shrinkage, so that the remaining errors can be treated as random errors and be adjusted as such through the least squares method.

V. FILM DISTORTION

A. Investigation

1. Theoretical Considerations

The correction of aerial photographs for film distortion is, in general, one of interpolation. A certain number of points in the photographic plate are used as control or known points, and from the information that they provide the images of other points will be displaced to locate them as close as possible to the position that they would occupy had there not been any distortion. As in any interpolation procedure, success depends on the number of known pieces of information available as well as on the knowledge of the general pattern that the system follows. For example, if measurements are made of the x and y coordinates of points on a parabola, we can, from the known properties of such a curve, make the following analysis:

The general analytical expression for a parabola with axis of symmetry parallel to the y axis is:

$$y = C_1 + C_2x + C_3x^2 \qquad (1)$$

in which the three C's are arbitrary constants. If in the above described manner the coordinates of three different points are measured, say points $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, $P_3(x_3,y_3)$, the following relations can be written by using (1).

$$\begin{aligned} y_1 &= C_1 + C_2x_1 + C_3x_1^2 \\ y_2 &= C_1 + C_2x_2 + C_3x_2^2 \\ y_3 &= C_1 + C_2x_3 + C_3x_3^2 \end{aligned} \qquad (2)$$

This provides a set of three equations and three unknowns which usually have a unique solution for the C's. If the measurements were carried in such a fashion that no errors are made, that is, they are exact (a physical impossibility), the three C's, when used in (1) will give the exact formula of the parabola. However, when the measurements are not exact, the C's will only give an approximate expression for the curve.

If more than three points on the parabola are measured, say n, a set of n equations in three unknowns is obtained in which any three of them will be solvable for the C's. There are $n!/3!(n-3)!$ different possible groups of three so that the same number of parabolas can be formed which will, in general, be different from each other. To determine from them what parabola to take, statistical methods should be used. The most suitable approach is that of minimizing the sum of the squares of the residuals.[1]

If, however, we did not know that the curve is a parabola, the problem could become much more complicated because either a guess of the type of curve will have to be made or else enough points will have to be measured in order to fit either a Fourier or an algebraic polynomial. When this is the case, a study of the physical characteristics of the problem can, in many instances, be very helpful in determining the exact function or at least an approximation to the exact function.

---

1. See F. B. Hildebrand, Introduction to Numerical Analysis, McGraw-Hill, 1956, for Least Squares Methods.

The above example is one of one-dimensional interpolation; that is, the unknown function depends on only one variable. Film distortion correction is, however, a two-dimensional interpolation problem since the distortion (or correction) at a point can, in general, depend on two independent variables (usually x and y). Two-dimensional interpolation is basically analogous to the one-dimensional case except that lengthier computations are involved, and in general more data is required for a satisfactory solution.

One further complication is present in the film distortion correction problem. At any one point, the distortion is not a scalar but rather a vector; that is, it has a direction as well as a magnitude associated with it. In the analysis that follows, it will be shown that the vector nature of the distortion does not complicate the problem appreciably.

As stated earlier, the problem of compensating for film distortion can be considered as one of defining a vector field. The vector field provides the size and the direction of the correction that should be applied to every point on the photograph. The same image results as existed when the film was exposed.

The first problem is to determine suitable coordinate axes. Obviously, whatever form the vector field has, it will be expressible in any coordinate system. The selection of a coordinate system is fairly arbitrary, and the system more suited to each phase of study should be used. Once the axes have been selected, every point P_i $(i = 1,2,......n)$ in a photograph can be located by the vector $V_k k$ which, in turn, can be expressed in terms of its X_k and Y_k components:

$$\bar{V}_k = \bar{V}_{kx} + \bar{V}_{ky} = X_k\bar{i} + Y_k\bar{j} \qquad (3)$$

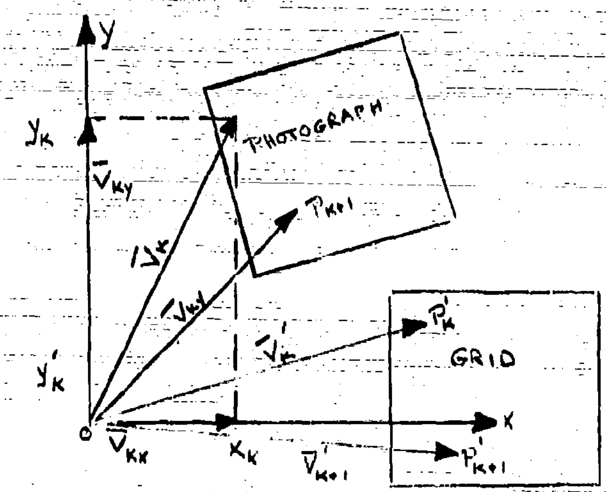

Figure 1

where $\overline{i}$ and $\overline{j}$ are unit vectors in the x and y direction respectively.

The absolute location of P_k in the X, Y plane is of no importance; the only thing of importance is the relative location of all the points.

If the convention is adopted that P_k be the points on the photograph and P_k' the corresponding points on the master grid that was used to make the prints, we can see that if by definition,

$$\overline{E}_k = \overline{V}_k' - \overline{V}_k \qquad (4)$$

a transformation of the set of points P_k of the form:

$$\overline{V}_k' = \overline{V}_k + \overline{E}_k \qquad (5)$$

will obviously define a set of points with the exact same geometrical relations as those of P_k. The set of vectors $\overline{E}_k$ can thus be said to represent the discrete value of a vector field $\overline{E} = \overline{F}$ (X,Y) at the points (X_k, Y_k). This vector field satisfies the necessary conditions at all points whose coordinates are known both on the film and on the grid. It should be noted, however, that an infinite number of vector fields take

take values E_k at P_k, and unless an infinite number of points are known, the true $\overline{E}$ cannot be singled out, unless other information is given. We do know, however, that $\overline{F}$ (X,Y) must be continuous unless rips or tears are present in the film.

In an empirical study like the one at hand, $\overline{E}$ will hopefully be determined by making sufficient observations at close enough intervals so that an interpolation will give it within the accuracy desired. As pointed out earlier, the interpolation problem is not an easy one because it is a two-dimensional problem and also because no information is yet available as to the shape of the distortion field.

The representation of $\overline{E}$ requires a four-dimensional space. There are two independent variables, X and Y, and two dependent ones, ie (magnitude of E) and (angle of E with the x-axis). It is found convenient to break the problem down into two parts, each dealing with a three-dimensional space in the following manner:

from equation (4)

$$\overline{E}_k = \overline{V}_k' - \overline{V}_k = (X_k'\overline{i} + Y_k'\overline{j}) - (X_k\overline{i} + Y_k\overline{j})$$

$$E_k = (X_k' - X_k)i + (Y_k' - Y_k)j \qquad (6)$$

letting:

$$e_{xk} = X_k' - X_k$$

$$e_{yk} = Y_k' - Y_k \qquad (7)$$

equation (4) can be rewritten:

$$\overline{E}_k = e_{xk}\overline{i} + e_{yk}\overline{j} \qquad (8)$$

In general, $\overline{E}$ is a function of X and Y. The X and Y components of $\overline{E}$ will thus be functions of X and Y:

$$\bar{E} = \bar{F}(X,Y) = f(X,Y)\bar{i} + g(X,Y)\bar{j} \qquad (9)$$

If it is desired that E agrees with $V_k' - V_k$ at the point P_k, the following relation must hold

$$\bar{E}_k = \bar{F}(X_k,Y_k) = f(X_k,Y_k)\bar{i} + g(X_k,Y_k)\bar{j} = e_{xk}\bar{i} + e_{yk}\bar{j}$$

from which:

$$f(X_k,Y_k) = e_{xk}$$

$$g(X_k,Y_k) = e_{yk} \qquad (10)$$

So, once f and g are known, e_x and e_y are obtained identically.

The problem has now reduced to the determination of two scalar point functions f(X,Y) and g(X,Y). Each of these functions can be visualized as a surface in three-dimensional space.

There being no information available as to the form of f and g, a form should be found that conforms best with the true deformation. This can only be done after an extensive empirical study.

The most popular interpolation formulas deal either with approximations by means of polynomials or by the use of trigonometric series.

Expressing f and g as algebraic polynomials, the following equations are obtained:

$$f(X,Y) = C_{xo} + C_{x1}X + C_{x2}Y + C_{x3}XY + \ldots + C_{xn}X^rY^s + e_x(X,Y)$$

$$g(X,Y) + C_{yo} + C_{y1}X + C_{y2}Y + C_{y3}XY + \ldots = C_{yn}X^rY^s + e_y(S,Y) \qquad (11)$$

where the C's are constants which will usually be determined from the available data.

At the points P_k where the E_k vector is known, the following relations can be written by combining equations (7), (10), and (11):

$$X_k' - X_k = C_{xo} + C_{x1}X_k + C_{x2}Y_k + \ldots\ldots$$

$$Y_k' - Y_k = C_{yo} + C_{y1}X_k + C_{y2}Y_k + \ldots\ldots \qquad (12)$$

In equation (12) the only unknowns are the C's. Each known point gives two equations, one for f and one for g. The maximum size of the interpolating polynomial, as well as the maximum number of constants in it, depends then on the number of available known points.

In photogrammetric practice, only four points of known position are available in each photograph. This limits the number of constants that can be solved for to four.

Some special cases of the polynomial interpolation are:

a) Uniform linear correction.

For this case, equation (11) reduces to:

$$f(X,Y) = C_{xo} + C_1 X$$
$$g(X,Y) = C_{yo} + C_1 Y \qquad (13)$$

b) Linear correction in X and Y

This is the correction usually applied in photogrammetric work. The procedure is to multiply all distances in the x-direction by the ratio $\frac{d_f}{d_g}$ where d_f is the distance between the related fiducial marks on the photograph and d_g is the same distance as measured in the master grid. In the notation of this report, this case becomes:

$$f(X,Y) = C_{xo} + C_{x1} X$$
$$g(X,Y) = C_{yo} + C_{y1} Y \qquad (14)$$

c) Linear interpolation

This differs from the previous case in that f and g are linear functions of both x and y. Thus,

$$f(X,Y) = C_{xo} + C_{x1} X + C_{x2} Y$$

$$g(XKY) = C_{yo} + C_{y1}X + C_{y2}Y \qquad (15)$$

d) Tewinkels's Formula

Mr. Tewinkel of U.S.C.G.S. suggests a correction formula of the form:

$$f(X,Y) = C_{xo} + C_{x1}X + C_{x2}Y + C_{x3}XY$$

$$g(X,Y) = C_{yo} + C_{y1}X + C_{y2}Y + C_{y3}XY \qquad (16)$$

The above formulas will be discussed and evaluated later in this report.

As a concluding remark in this section, it should be noted that, although algebraic polynomial interpolation is the more popular type of correction, there is yet no evidence that it is the best one for our purpose. Furthermore, the above-mentioned polynomials are not the only possible ones; for example $(X^2 + Y^2)$ or X^2 and Y^2 can be used instead of XY in the last formula. A trigonometric or, in general, a transcendental equation might give better results than an algebraic one. The decision as to which would be the more efficient type of equation can only be made by a statistical evaluation of a large group of test data.

2. Tests on the Different Interpolation Methods

Since this laboratory has no facilities for the preparation of data for a film distortion study, we had to rely on data from outside sources.

The only data that we could obtain which seemed to be of any value for the project, was a set of four photographs prepared by the U.S. Coast and Geodetic Survey. This data consisted of the comparator coordinates of thirteen grid intersections from a master plate and the contact prints made from it (on aerographic film). The time of measurements ranged between five days to forty-seven days after development.

This data was used to make a statistical evaluation of the interpolation methods discussed in the previous section.

A set of programs were run on the IBM 1620. In these programs, all four methods were used for the interpolation, and the constants in each formula were obtained once by using the corner fiducials as controls and once using the side fiducials as controls. After obtaining the appropriate constants, the film coordinates were used in the formulas and the corrected image coordinates were compared to the plate coordinates. The sum of the squares of the remaining errors in the X and Y of the corrected image coordinates gave an indication of the effectiveness of the different methods. It should be noted here that theoretically, in the first three methods, the absolute locations of the origin or the orientation of the coordinate axes do not affect the corrections. In Mr. Tewinkel's method, however, although the absolute location is not significant, the orientation of the coordinate system does affect the corrections. This is evident when one considers that, in this method, the curves of constant correction are hyperbolas with asymptotic directions parallel to the X and Y axes; so as these curves of constant correction depend on the orientation of the axes, the corrections also depend on the orientation of the axes. It should also be noted that there are polynomials whose corrections would depend on the absolute location of the origin and the orientation of the coordinate system. A simple example of this case is:

$$f(X,Y) = C_x X$$

$$g(X,Y) = C_y Y$$

Keeping the above considerations in mind, the following tests were made:

(1) methods (a)[1], (b), and (c) once with the corners as controls and once with the sides as controls; (2) method (d) with origins outside the photograph, once using the corners as controls and once using the sides as controls; (3) method (d) with origin at the center of the photograph and axes almost parallel to the sides of the photograph, once using the corners as controls and once using the sides as controls; (4) method (d) with origins at the center and the axes almost along the diagonals of the photograph, once using the corners as controls and once using the sides as controls.

In methods (a), (b), and (c), as there were more equations than unknowns, normal equations were formed and the resulting set of equations were solved by the Jordan diagonalization method. The significance of using two different locations for the origins for method (d) will be discussed in the next section. In the following table, the results of these runs on three sets of data are listed, using the sum of the squares of the errors in X and Y of the corrected images as a measure of the efficiency of each method.

1. (a), (b), and (c) refer to the methods described in the last section.

METHOD OF INTERPOLATION	SUM SQUARE OF X AND Y ERRORS FOR THIRTEEN POINTS IN (MICRONS)²		
	DATA # 1	DATA # 2	DATA # 3
(a) WITH CORNERS AS CONTROLS	7349	8242	7232
(a) WITH SIDES AS CONTROLS	7660	8539	7078
(b) WITH CORNERS AS CONTROLS	5489	7415	7116
(b) WITH SIDES AS CONTROLS	6315	7841	7142
(c) WITH CORNERS AS CONTROLS	5075	2667	3015
(c) WITH SIDES AS CONTROLS	5402	3109	3034
(d) WITH ORIGIN OUTSIDE AND AXES PARALLEL TO THE SIDES OF THE PHOTOGRAPH AND USING THE CORNERS AS CONTROLS.	1351	915	736
(d) WITH ORIGIN OUTSIDE AND AXES PARALLEL TO THE SIDES OF THE PHOTOGRAPH USING THE SIDES AS CONTROLS.	10^9	2.1×10^7	3.6×10^7
(d) WITH ORIGIN AT CENTER AND AXES PARALLEL TO THE SIDES AND CORNERS AS CONTROLS.	1351	915	736
(d) WITH ORIGIN AT CENTER AND AXES PARALLEL TO THE SIDES AND SIDES AS CONTROLS.	4×10^{10}	2.6×10^9	2.4×10^9
(d) WITH ORIGIN AT CENTER AND AXES ACROSS THE DIAGONALS OF THE PHOTOGRAPH AND THE CORNERS AS CONTROLS.	1.3×10^{10}	1.9×10^9	3×10^9
(d) WITH ORIGIN AT CENTER AND AXES ACROSS THE DIAGONALS OF THE PHOTOGRAPH AND THE SIDES AS CONTROLS	4907	2989	2936

TABLE SHOWING THE SUM SQUARE OF ERRORS IN X AND Y OF THE CORRECTED COORDINATES OF THIRTEEN POINTS (TWENTY-SIX COORDINATES), FOR DIFFERENT METHODS OF INTERPOLATION.

(a), (b), (c), AND (d), REFER TO METHODS DESCRIBED IN THE LAST SECTION.

B. <u>Discussion</u>

Although three sets of data are not sufficient for a reliable statistical evaluation of the different methods of interpolation, one can still expect that the distortion pattern of these three photographs after correction would be indicative of the efficiency of each method. Furthermore, one can expect a certain method to be preferable to another by considering some theoretical aspects.

Methods (a) and (b) are rather inefficient, and the sum square of the errors in our three photographs confirm this fact. These two methods gave the largest errors in every case, whether using the corner fiducials or the side fiducials as controls. Method (c) gave better results than methods (a) and (b). Although in method (c) the corner fiducials gave better results than the side fiducials in all three cases, the difference was not significant enough to indicate that corner fiducials are better than side fiducials for this method.

Method (d) provided more problems than the first three methods. Although it was deduced that the absolute location of the origin should not affect the results of the interpolation, when the axes were parallel to the sides of the photograph and the side fiducials were used as controls, the results did depend on the location of the origin. It It should be noted that, in these cases, there was an enormous amount of errors left after correction. Although these large errors were not all due to round-off error, there was evidence that there was a large amount of round-off error in the case where the origin was outside the photograph with the axes parallel to the sides of the photograph and the side fiducials used as controls. This was evident from the fact that, when the corrections of the

control points themselves were obtained, they had errors up to ten microns. This suggests that the difference of the results due to change of origin was due to a different amount of round-off error in each case.

The large error associated with the above case was due to using "weak" points as controls and the greatest amounts of error were associated with points whose XY products were large. We noted earlier that, when using M . Tewinkel's formula, the curves of constant correction are hyperbolas. Considering the X correction, C_{xo}, C_{x1}, C_{x2}, and C_{x3} define a family of hyperbolas with the same center and asymptotes. Each hyperbola is defined as a function of f or the x correction, and the magnitude of f increases as the hyperbolas become farther from the center. (Refer to Figure 2.)

When the axes were rotated forty-five degrees to make them almost along the diagonals of the photograph; as should be expected, the side fiducial gave good results, whereas the corner fiducials gave ery large errors for points with large XY (origin at center of photograph). This is, of course, exactly the reverse of the case where the axes are parallel to the sides.

Note: When solving for the constants of Tewinkel's formula, it is theoretically possible that the determinant of the matrix involved will be zero and no solution will exist. Such a case will arise when there is absolutely no error in the coordinates of the fiducials on the film with respect to the fiducials on the grid and the lines joining any two fiducials on the film are parallel to lines joining the same fiducials on the grid. Although such a case is theoretically possible, it is highly improbable that it will occur, as there will always be some kind of error present in the coordinates of the fiducials.

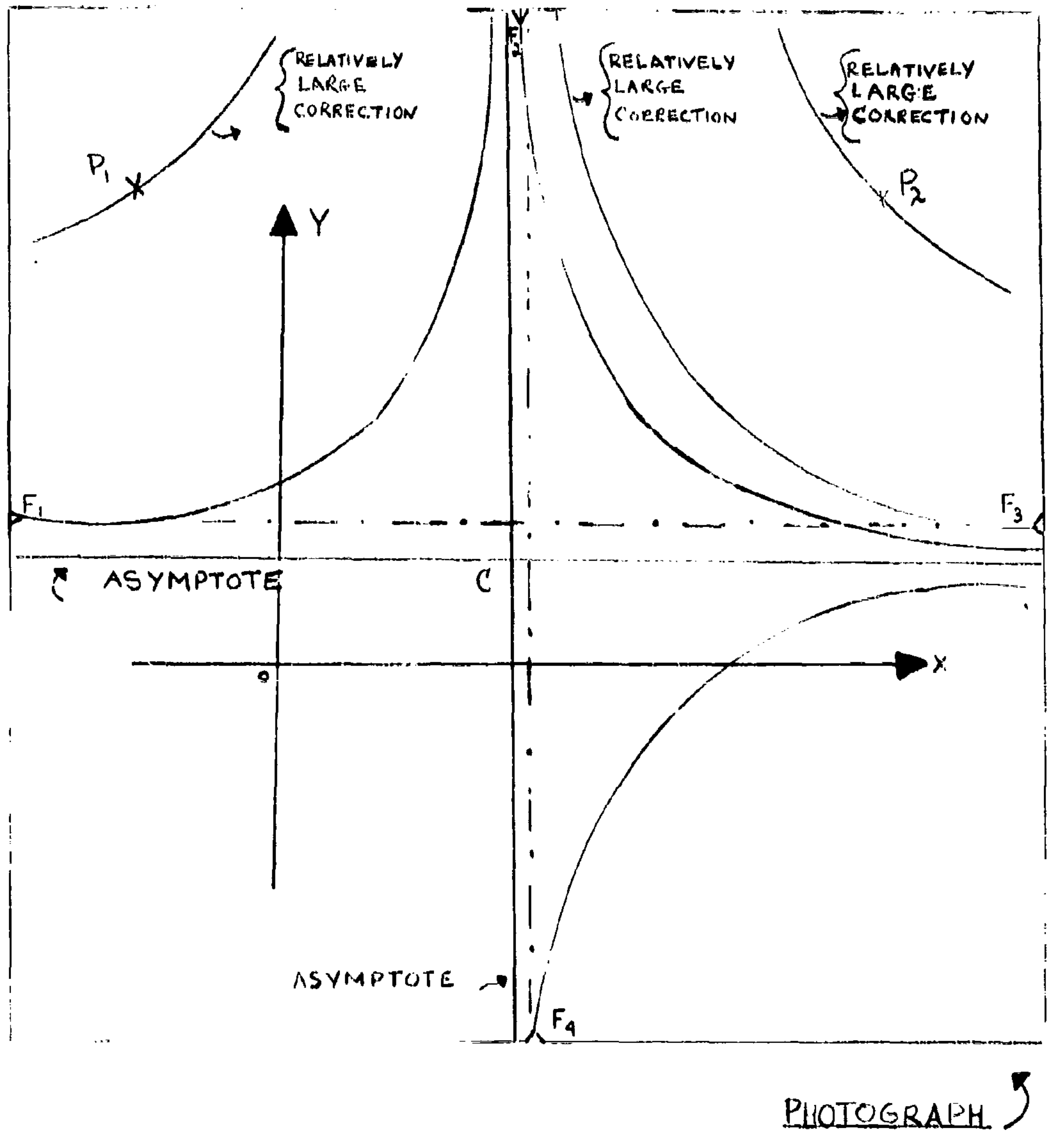

Figure 2: This diagram shows a typical distortion correction pattern when Tewinkel's method is used with side fiducials F_1, F_2, F_3, F_4 as controls and X, Y axes parallel to the sides of the photograph. C is the center of the family of hyperbolas and all points on a specific hyperbola will have the same correction. P_1 and P_2 will have unnecessarily large corrections.

Another point that should be considered is that if the determinant of the matrix is not exactly zero, but close to zero, there will be large errors in the solutions. Such a case can, in general, be detected by computing the remaining errors after correction in the coordinates of the fiducials themselves. Now, in all three cases tried the center of the family of hyperbolas was close to the center of the photograph, and so the side fiducials or our controls were all close to one or the other asymptote. On the other hand, it was observed that the corner points of the photographs usually had the same order of absolute magnitude of error before correction as the side points.

So, when these side fiducials which are close to the asymptotes are used to obtain the family of hyperbolas, a point which is far from both asymptotes, e.g., a corner point will be associated with a very large f, whereas the true f might be the same order of magnitude as for a side point. Therefore, when side fiducials are used as controls, unnecessarily large amounts of correction are made on all points whose XY are large and, hence, poor interpolation results are obtained.

On the other hand, when corner fiducials are used as controls and the axes are parallel to the sides of the photograph, all the fiducials cannot be close to the asymptotes; therefore, the XY of the corner points being usually the largest of any point on the photograph, the family of hyperbolas are so defined that no point on the photograph will be associated with an unusually large correction and so better results are obtained.

C. Conclusions

1. The data available was insufficient to draw any finalized conclusions.

2. As film distortion is nearly a random process, there is a limit to the accuracy of any interpolation method that may be used. On the average, this accuracy will increase as the number of control points becomes greater.

3. Defining $>$ as "is better than", the data available has indicated that:

Tewinkel's formula $>$ Linear interpolation $>$ Linear correction in X and Y $>$ Uniform linear correction.

4. The corner fiducials gave better results than side fiducials in all tests run on the three sets of data available. However, in methods (a), (b), and (c) there was not any significant difference between the results to indicate a general pattern.

5. When using Tewinkel's formula, to assure reliable results the axes should be parallel to the sides of the photograph when the corner fiducials are used, and the axes should be parallel to the diagonals when the side fiducials are used. Furthermore, there seems to be less round-off error when the center of the photograph is used as the origin.

VI. THE SOLUTION OF SIMULTANEOUS EQUATIONS

A. Introduction

A general study of different methods of solutions of simultaneous equations was made to determine whether other methods might be more efficient than the Stifel method of iteration for the solution of the normal matrix of aerotriangulation.

As a result of this study, the Jordan diagonalization method has been programmed for the object program and in its final form can be used when there are less than 25 photographs in the aerotriangulation problem. This is discussed further in Appendix D.

We shall use the following notation for the set of simultaneous equations that are to be solved.

$$|A|\cdot|X| = \begin{vmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N-1} & A_{1,N} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,N-1} & A_{2,N} \\ \vdots & \vdots & & \vdots & \vdots \\ A_{N,1} & A_{N,2} & \cdots & A_{N,N-1} & A_{N,N} \end{vmatrix} \cdot \begin{vmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{vmatrix} = \begin{vmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{vmatrix} = |C|$$

$A_{i,j}$'s are the equation coefficients, x_i's are the unknowns, and c_i's are the constants associated with the equations.

B. Direct Methods

1. Jordan Diagonalization Method

a. Restrictions:

Matrix $|A|$ must have non-zero diagonal elements. The program will stop if any of the diagonal elements become zero during the

```
RM     JORDAN DIAGONALIZATION METHOD
RM     FOR SOLUTION OF SIMULTANEOUS EQUATIONS
RM     650 BUFFTRAN SOURCE PROGRAM
       DIMEN A(27,28)
RM     N MUST BE LESS THAN 28
12     READ,N
       M=N+1
       DO 1 I=1,N
       DO 1 J=1,M
 1     READ, A(I,J)
       DO 4 I=1,N
       R=A(I,I)
       J=I+1
       DO 13 L=J,M
13     A(I,L)=A(I,L)/R
       DO 4 K=1,N
       IF(K-I)8,4,8
 8     P=A(K,I)
       DO 14 L=J,M
14     A(K,L)=A(K,L)-A(I,L)*P
 4     CONTINUE
       DO 9 J=1,N
 9     PUNCH, A(J,M)
       GO TO 12
       END
```

solution. This is, in general, quite likely to happen.

b. Input Format:

First card contains N, the number of equations, then the elements of matrix A and vector C are read in, one word per card, in the following order:

$A_{1,1}$, $A_{1,2}$, ... $A_{1,n-1}$, $A_{1,n}$, C_1, $A_{2,1}$, $A_{2,2}$, ... $A_{2,n-1}$, $A_{2,n}$, C_2, ... $A_{n,1}$, $A_{n,2}$, ... $A_{n,n-1}$, $A_{n,n}$, C_n.

c. Output Format:

Solutions to the unknowns X_i are punched one word per card in the following order: X_1, X_2, ... X_n.

d. Running Time:

The program takes about 50 seconds on the IBM 650 when N, the number of equations, is four. Forty-five seconds are required on the IBM 1620 for N equal to eleven.

2. <u>Jordan Diagonalization Method for Symmetric Matrix</u>

This is essentially the same as the previous program except that it makes use of the symmetry of matrix A and has fewer operations.

a. Restrictions:

Matrix A must be symmetric and must have non-zero diagonal elements. The program will stop if a diagonal element becomes zero during the solution. This is, in general, quite likely to happen.

b. Input Format:

The first card contains N, the number of equations. Then

```
RM     JORDAN METHOD FOR SOLUTION OF A SYMMETRIC MATRIX.
RM     650 BUFFTRAN SOURCE PROGRAM
       DIMEN A(24,25),B(25)
RM     N MUST BE  LESS THAN 25
12     READ,N
       M=N+1
       DO 1 I=1,N
       DO 1 J=1,M
 1     READ,A(I,J)
       DO 4 I=1,N
       J=I+1
       R=A(I,I)
       DO 13 L=J,M
       B(L)=A(I,L)
13     A(I,L)=A(I,L)/R
       DO 4 K=1,N
       IF(K-I)8,4,5
 8     P=A(K,I)
       DO 14 L=J,M
14     A(K,L)=A(K,L)-A(I,L)*P
       GO TO 4
 5     P=B(K)
       DO 7 L=K,M
 7     A(K,L)=A(K,L)-A(I,L)*P
 4     CONTINUE
       DO 9 J=1,N
 9     PUNCH,A(J,M)
       GO TO 12
       END
```

the elements of matrix A and vector C are read in one word per card in the following order:

$A_{1,1}, A_{1,2}, \ldots A_{1,n-1}, A_{1,n}, C_1, A_{2,1}, A_{2,2} \ldots$
$A_{2,n-1}, A_{2,n}, C_2, \ldots A_{n,1}, A_{n,2}, \ldots A_{n,n-1},$
$A_{n,n}, C_n$

c. Output Format:

Solutions to the unknown X_i are punched one word per card, in the following order: $X_1, X_2, \ldots X_n$.

d. Running Time:

The program takes about 40 seconds on the IBM 650 when the number of equations is four. No other estimate is available, but it is faster than the general Jordan Diagonalization method discussed previously.

3. <u>Inverse Matrix Method</u>

This program is not very efficient and is <u>not</u> particularly recommended if the number of equations is large. It is being mentioned here as it was programmed in the preliminary studies and may be used to compute the inverse of a matrix.

a. Restrictions:

The diagonal elements must be non-zero.

b. Input Format:

First card contains N, the number of equations. Then the elements of matrix A and vector C are read in in the following order:

```
RM     SOLUTION OF SIMULTANEOUS EQUATIONS
RM     BY INVERSE MATRIX METHODS
RM     650 BUFFTRAN SOURCE PROGRAM
       DIMEN A(12,12),B(12,12),C(12)
     1 X(12)
12     READ,N
RM     N MUST BE LESS THAN 13
       M=N+1
       DO 1 I=1,N
       DO 1 J=1,N
       READ, A(I,J)
       IF(I-J)2,3,2
 2     B(I,J)=0.0
       GO TO 1
 3     B(I,J)=1.0
 1     CONTINUE
       DO 15 J=1,N
 15    READ, C(J)
       DO 4 I=1,N
       R=A(I,I)
       DO 13 L=1,N
       B(I,L)=B(I,L)/R
 13    A(I,L)=A(I,L)/R
       DO 4 K=1,N
       P=A(K,I)
 7     IF(K-I)8,4,8
 8     DO 14 L=1,N
       B(K,L)=B(K,L)'B(I,L)*P
 14    A(K,L)=A(K,L)'A(I,L)*P
 4     CONTINUE
       IF(M-2.0)10,11,10
 10    DO 9 I=1,N
       DO 9 J=1,N
 9     PUNCH, B(I,J)
 11    DO 5 I=1,N
       X(I)=0.0
       DO 5 J=1,N
 5     X(I)=B(I,J)*C(J)+X(I)
       DO 6I=1,N
 6     PUNCH,X(I)
       GO TO 12
       END
```

$A_{1,1}, A_{1,2}, \ldots, A_{1,n-1}, A_{1,n}, A_{2,1}, A_{2,2}, \ldots$
$A_{2,n-1}, A_{2,n}, \ldots, A_{n,1}, A_{n,2}, \ldots, A_{n,n-1},$
$A_{n,n}, C_1, C_2, \ldots, C_n.$

c. Output Format:

If we denote the elements of the inverse matrix of A by $B_{i,j}$, then the first N^2 cards contain the elements of the inverse in the following order:

$B_{1,1}, B_{1,2}, \ldots, B_{1,n-1}, B_{1,n}, B_{2,1}, B_{2,2}, \ldots$
$B_{2,n-1}, B_{2,n}, \ldots, B_{n,1}, B_{n,2}, \ldots, B_{n,n-1},$
$B_{n,n}$

The last N cards contain the unknown X_i in the following order: $X_1, X_2, \ldots X_n$.

d. Running Time:

The program takes about one minute and 10 seconds on the IBM 650 for N equal to four.

C. <u>Iterative Methods</u>

1. <u>Seidel Method of Iteration</u>

a. Restrictions:

Matrix A has to be symmetric and positive definite. A set of normal equations has these properties, and therefore it is possible to use this method to solve the normal matrix of the aerotriangulation problem.

b. Input Format:

To use this program matrix A and the constant, vector C should have already been stored in the computer. Proper storage can be achieved by using the program for formation of normal

```
RM     SIEDEL S METHOD OF ITERATION FOR A SYMMETRIC
RM     MATRIX TO BE USED ALONG AND AFTER THE
RM     PROGRAM FOR FORMATION OF NORMAL EQUATIONS
RM     650 BUFFTRAN SOURCE PROGRAM
       DIMEN A(18,19),Y(18),X(18)
RM     N MUST BE LESS THAN 19
       READ,N,R,P
       L=N+1
       DO 3 I=1,N
 3     Y(I)=A(I,L)
       IF(R-5.)5,6,6
 6     DO 8 I=1,N
 8     READ,X(I)
       GO TO 20
 5     DO 9 I=1,N
 9     X(I)=0.
20     M=1.
       H=0.
10     DO 12 K=1,N
       G=0.
       DO 11 I=1,N
11     G=G+A(K,I)*X(I)
       Q=(Y(K)-G)/A(K,K)
       X(K)=X(K)+Q
       IF(K-M)12,13,12
13     IF(Q-P)14,14,12
14     M=M+1
       IF(M-N)12,12,17
12     CONTINUE
       H=H+1
       GO TO 10
17     DO 18 I=1,N
18     PUNCH,X(I)
       PUNCH,H
       STOP
       END
```

equations described below.

This program itself reads at least one card on which there should be three values designated by N, R, and P respectively. N is the order of matrix A . If R 5.0, it will be assumed that there are no initial approximations available to the unknowns, and the program will start from initial approximations of zero to the unknowns and no more cards will be read. If R 5.0, it will be assumed that there are some initial approximations available to the unknowns, and the program will read N cards containing the approximations to X_1, X_2, ... X_n respectively. P should be in the order of magnitude of the maximum error permissible in any of the unknown.

c. Output Format:

When the error in all the unknowns is in the order of magnitude of P (described above), N+1 cards will be printed out. The first N cards will contain the unknowns X_1, X_2, ... X_n respectively. The last card will contain the number of iterations the program has gone through.

d. Running Time:

Running time depends entirely on the matrix A , the initial approximations used, and the accuracy desired.

This program is, in general, efficient only for improving the results of a direct method of solution.

2. <u>Stiefel Method of Iteration with Dirac Density Function</u>

a. Restrictions:

Matrix A has to be symmetric and positive definite. It

```
RM      STIEFEL METHOD OF ITERATION WITH DIRAC DENSITY FUNCTION
RM      TO BE USED ALONG AND AFTER NORMAL EQUATION PROGRAM
RM      650 BUFFTRAN SOURCE PROGRAM
        DIMEN A(18,19),X(18),DPX(18),R
    1   (18),DPR(18),AR(18)
RM      N MUST BE LESS THAN 19
        READ,N,NUM
        L=N+1
        Q=0.
        CP=1.
        DO 6 I=1,N
        X(I)=0.
        DPX(I)=0.
        R(I)=A(I,L)
  6     DPR(I)=0.
  3     H=0.
  2     H=H+1
        DO 7 I=1,N
        AR(I)=0.
        DO 7 J=1,N
  7     AR(I)=AR(I)+A(I,J)*R(J)
        C=0.
        DO 8 I=1,N
  8     C=C+AR(I)*R(I)
        CA=0.
        DO 9 I=1,N
  9     CA=CA+AR(I)*AR(I)
        P=(C*Q)/CP
        Q=CA/C-P
        DO 12 I=1,N
        DX=(R(I)+P*DPX(I))/Q
        X(I)=X(I)+DX
        DPX(I)=DX
        DR=(P*DPR(I)-AR(I))/Q
        R(I)=R(I)+DR
 12     DPR(I)=DR
        CP=C
        IF(H-NUM)2,4,4
  4     DO 5 I=1,N
  5     PUNCH,X(I),R(I)
        STOP
        END
```

should be noted that a set of normal equations has these properties.

b. Input Format:

To use this program, matrix A and the constant vector C should have already been stored in the computer. Proper storage can be achieved by using the program for formation of normal equations described below.

This program itself reads one card containing the two values N and NUM respectively. N is the order of matrix A , NUM is the number of iterations desired - it should in general be a multiple of N.

c. Output Format:

When the program has gone through the desired number of iterations, it will punch N cards, each containing two values. The first value on the ith card will be the unknown x_i, and the second value will be the residual associated with the ith equation.

d. Running Time:

Running time depends on the accuracy desired and, consequently, on the number of iterations for four equations. Each set of four iterations takes about thirty seconds on the IBM 650. For four equations, at least eight iterations were necessary to obtain an accuracy of 10^{-7}.

3. <u>Stiefel Method of Iteration Using Chebyshev and Dirac Density Funtions</u>

This method is essentially the same as program 8,2 discussed above, the only difference being that it computes some initial approx-

imations to the unknowns before starting that program. This process consists of three separate programs which are to be used one after the other. Data common to more than one program is permanently stored and carried on from one program to another by proper dimensioning.

4. Determination of the Smallest and Largest Eigenvalues of Matrix A

This program uses a very crude method for the determination of the largest eigenvalue of matrix A and then takes an assigned fraction of it as the smallest eigenvalue.

a. Restrictions

Matrix A has to be symmetric and positive definite.

b. Input Format:

To use this program, matrix A and the constant vector C should have already been stored in the computer. Proper storage can be achieved by using the program for fo mation of normal equations, described below.

This program itself reads one card containing two values denoted by N and C respectively. N is the order of matrix A . C is the fraction of the largest eigenvalue that is supposed to be considered as the smallest eigenvalue. A value of C = 0. 4 or 0.02 usually can give a reasonable approximation to the smallest eigenvalue. The better this approximation is, the better the final results of solution will be.

c. Output Format:

This program prints out one card containing the computed maximum and minimum eigenvalues respectively.

```
RM      MAXIMUM MINIMUM ROUTINE
RM      PROGRAM FOR DETERMINATION OF
RM      THE GREATEST AND SMALLEST
RM      EIGENVALUES OF A GIVEN MATRIX
RM      TO BE USED AFTER THE PROGRAM FOR NORMAL
RM      EQUATION FORMATION AND BEFORE STIEFEL
RM      650 BUFFTRAN SOURCE PROGRAM
        DIMEN A(18,19),X(18),Y(18),
    1   NN(1),MAXA(1),MINA(1)
RM      N MUST BE LESS THAN 19
        READ,N,C
        MAX=0.
        DO 1 I=1,N
        MIN=0.
        DO 2 J=1,N
  2     MIN=MIN+A(I,J)
        IF(MAX-MIN)5,1,1
  5     MAX=MIN
  1     CONTINUE
        MIN=C*MAX
        MAXA(1)=MAX
        MINA(1)=MIN
        NN(1)=N
        PUNCH,MAX,MIN
        STOP
        END
```

d. Running Time:

Running time is around five seconds on the IBM 650 for four equations.

5. Stiefel Method of Iteration with Chebyshev Density Function

This method of iteration by itself is not very efficient and is used only to compute initial approximations for the Stiefel method with Dirac Density Function which will be explained later.

a. Restrictions:

Same as above.

b. Input Format:

This program reads only one card containing the number of iterations desired. This number should in general be a multiple of N. The matrix A and vector C should already have been stored. Proper storage will be achieved if the program above has been used correctly.

c. Output Format:

This program does not have any output. Relevant information is stored in memory to be used by the program next described.

d. Running Time:

An accurate running time is not available. But, it is expected that for N equal to four, using four iterations, about thirty seconds would be required on the IBM 650.

6. Stiefel Method of Iteration with Dirac Density Function

Except for the information obtained and used from the program above, this program is the same as program C,2.

```
RM      STIEFEL METHOD OF ITERATION WITH CHEBYSHEV
RM      DENSITY FUNCTION TO BE USED AFTER THE MAX
RM      MIN ROUTINE AND BEFORE STIEFEL DIRAC DENSITY
RM      650 BUFFTRAN SOURCE PROGRAM
        DIMEN A(18,19),X(18),Y(18),
    1   NN(1),MAXA(1),MINA(1),
    2   DPX(18),R(18)
RM      NUM MUST BE LESS THAN 19
        READ,NUM
        MAX=MAXA(1)
        MIN=MINA(1)
        N=NN(1)
        L=N+1
        MMM=MAX-MIN
        MPM=MAX+MIN
        BAR=MPM+2*SQRT(MAX*MIN)
        BARD=BAR/MMM
        BARD1=BARD
        RABD=MMM/BAR
        RABD1=RABD
        DO 1 I=1,N
        Y(I)=A(I,L)
        R(I)=Y(I)
        DPX(I)=2*R(I)/MPM
  1     X(I)=DPX(I)
        CSHNP=MPM/MMM
        COSHN=1.
        H=0.
  6     H=H+1
        DO 3 I=1,N
        YESTI=0.
        DO 5 J=1,N
  5     YESTI=YESTI+A(I,J)*X(J)
  3     R(I)=Y(I)-YESTI
        CSHNM=COSHN
        COSHN=CSHNP
        BARD1=BARD1*BARD
        RABD1=RABD1*RABD
        CSHNP=(BARD1+RABD1)/2
        DO 2 I=1,N
        DPX(I)=(4*COSHN*R(I)/MMM+CSHNM
    1   *DPX(I))/CSHNP
  2     X(I)=X(I)+DPX(I)
        IF(H-NUM)6,7,7
  7     STOP
        END
```

```
RM     STIEFEL ITERATION WITH DIRAC DENSITY
RM     TO BE USED AFTER THE STIEFEL WITH
RM     CHEBYSHEV DENSITY FUNCTION
RM     650 BUFFTRAN SOURCE PROGRAM
       DIMEN A(18,19),X(18),Y(18),NN(
     1 1),DPX(18),R(18),DPR(18),
     2 AR(18)
RM     NUM MUST BE LESS THAN 19
       READ,NUM
       N=NN(1)
       L=N+1
       Q=0.
       CP=1.
       DO 6 I=1,N
       DPR(I)=0.
       DPX(I)=0.
       YESTI=0.
       DO 11 J=1,N
 11    YESTI=YESTI+A(I,J)*X(J)
 6     R(I)=Y(I)-YESTI
 3     H=0.
 2     H=H+1
       DO 7 I=1,N
       AR(I)=0.
       DO 7 J=1,N
 7     AR(I)=AR(I)+A(I,J)*R(J)
       C=0.
       DO 8 I=1,N
 8     C=C+AR(I)*R(I)
       CA=0.
       DO 9 I=1,N
 9     CA=CA+AR(I)*AR(I)
       P=(C*Q)/CP
       Q=CA/C-P
       DO 12 I=1,N
       DX=(R(I)+P*DPX(I))/Q
       X(I)=X(I)+DX
       DPX(I)=DX
       DR=(P*DPR(I)-AR(I))/Q
       R(I)=R(I)+DR
 12    DPR(I)=DR
       CP=C
       IF(H-NUM)2,4,4
 4     DO 5 I=1,N
 5     PUNCH,X(I),R(I)
       STOP
       END
```

a. Restrictions:

Same as a. above

b. Input Format:

This program reads only one card containing the number of iterations desired. This number should in general be a multiple of N. Matrix A and vector C would already have been stored properly if the two previous programs have been run correctly.

c. Output Format:

When the program has gone through the desired number of iterations, it will punch N cards containing two values. The first value on the ith card will be the unknown X_i and the second value will be the residual associated with the ith equation.

d. Running Time:

Running time is about thirty seconds on the IBM 650 for N equal to four and four iterations.

D. Formation of Normal Equations

This program can be used to form a set of normal equations from a redundant or non-redundant set of equations. It will store the resultant set of normal equations in the proper storage space required by several of the previously described programs.

We shall consider the set of possibly redundant number of equations in the following form:

```
RM     FORMATION OFNORMAL EQUATIONS
RM     650 BUFFTRAN SOURCE PROGRAM
RM     N MUST BE LESS THAN 19
RM     M MUST BE LESS THAN 26
       DIMEN A(18,19),B(25,19)
       READ,N,M
       L=N+1
       DO 1 I=1,M
       DO 1 J=1,L
  1    READ,B(I,J)
       DO 5 I=1,N
       DO 5 J=1,L
       A(I,J)=0.
       DO 5 K=1,M
  5    A(I,J)=A(I,J)+B(K,J)*B(K,I)
       GO TO CONSW
       END
```

$$\begin{matrix} B_{1,1} & B_{1,2} & \cdots & B_{1,N-1} & B_{1,N} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,N-1} & B_{2,N} \\ \vdots & \vdots & & \vdots & \vdots \\ B_{M,1} & B_{M,2} & \cdots & B_{M,N-1} & B_{M,N} \end{matrix} \cdot \begin{matrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{matrix} = \begin{matrix} C_1 \\ C_2 \\ \vdots \\ C_M \end{matrix}$$

a. Input Format:

This program first reads one card containing N and M, then reads M x (N+1) cards containing the following values respectively:

$B_{1,1}$, $B_{1,2}$, ... , $B_{1,N-1}$, $B_{1,N}$, C_1, $B_{2,1}$, $B_{2,2}$, ... $B_{2,N-1}$, $B_{2,N}$, C_2, ... , $B_{M,1}$, $B_{M,2}$, ... , $B_{M,N-1}$, $B_{M,N}$, C_M.

b. Output Format:

This program does not have any output. It just stores the resultant matrix and constant vector in the following way. If we denote the resultant set of equations by out previous notation of matrix A and vector C , then the element $A_{i,j}$ will be in the location specified by A(i,j) in Fortran. The component C_i of vector C will be in location specified by A(i,N+1) in Fortran.

c. Running Time:

This program takes roughly 20 seconds on the IBM 650 when both M and N are four.

E. Remarks

1. All the programs discussed above were written solely for experimental purposes. They were essentially intended to be used with

a small number of equations; and, therefore, their input and output formats are in no way claimed to be the most efficient possible.

2. It should be noted that in IBM 650 Bufftran, which was the coding language used, all quantities are considered to be floating point quantities. Therefore, it might be necessary to change some of the notations in the programs if they are to be run on a standard Fortran system.

F. Conclusions

1. The Jordan Diagonalization and the Crout methods, which are quite similar, appear to be the simplest and fastest direct methods for solution of simultaneous equations.

2. The inverse matrix method is not very efficient because of its length and its inaccuracy.

3. The single step and total step iterations are not very efficient because of the restrictions for convergence.

4. The Seidel method of iteration for symmetric and positive definite matrices converges rapidly to a high accuracy when a good initial approximation is used. However, convergence is rather slow when poorer initial approximations are used although convergence is guaranteed. Therefore, in cases where the set of simultaneous equations have been solved by a direct method and the results are in error due to round-off, the Seidel method may be used to improve the results.

5. The Stiefel method with Dirac's Density Function converges quite rapidly for all matrices tested, and after 2N iterations, N being the number of equations and in this case 4 or 8, the maximum

relative error was in the order of 10^{-7}.

6. The Stiefel method with Chebyshev Density Function proved to give an initial approximation for the Stiefel method with Dirac's Density and eventually gave a better result than when an initial approximation of zero was used. But, this improvement still was in the order of 10^{-7} which suggests that the Stiefel method with Chebyshev Density might not be necessary at all.

7. Although the accuracy achieved by a direct method might take a longer time by an iterative method, in general iterative methods have an advantage as far as storage space is concerned. This is because iterative methods in general do not operate on the matrix of simultaneous equations, i.e., they do not change it. Whereas direct methods operate on the matrix, and non-zero elements might be created where initially there were zeros which did not require storage allocation.

8. The accuracy achieved by the Jordan Diagonalization method definitely requires a considerably longer time to be achieved by the Stiefel method.

A Modified Segment 3 has been written to be used when the higher accuracy possible with the Stiefel method is not required and 24 or less photographs are involved so that computer time may be conserved. The realized reduction in running time has been essential to the execution of this project. The Modified Segment 3 also represents a practical implementation of the solution of the aerotriangulation problem normal matrix.

VII. MATHEMATICAL DESCRIPTION OF THE PROGRAM

A. PROGRAM FUNCTION

Given a certain amount of photographic, exposure station, and ground point information, the program finds a simultaneous solution to the orientation and location of the exposure stations with regard to geocentric axes. This simultaneous solution is based on a least square adjustment of the random errors associated with the measured data.

B. INPUT DATA

The input data consists of parts of the following:

1. The measured x and y photographic coordinates of the images of ground points.
2. The geographic coordinates (latitude, longitude, elevation) of ground points and exposure stations.
3. Estimates of the geographic coordinates and orientation of the exposure stations.
4. Focal length of the camera, major and minor axes of the ellipsoid defining the geoid.

C. THE EQUATIONS

They can be divided into two groups:

1. Ground point equations

   These equations express either the:

   a. intersection of rays corresponding to the images of a ground point; or

   b. enforcement, when they are known, of the geographic coordinates of a ground point.

2. Exposure station equations

These equations enforce, when they are known, the geographic coordinates of an exposure station.

D. UNKNOWNS

These are of two types:

1. Exposure station unknowns

These are the only unknowns required for the solution of the photogrammetric problem. There are six unknowns associated with every exposure station. Three are corrections to the estimated geocentric coordinates of the perspective center (dRx, dRy, dRz), and the remaining three are the components along the geocentric axis of a differential rotation vector $\overline{dT}$ which is the correction to the orientation of the camera (dT_x, dT_y, dT_z).

2. Residual unknowns

These unknowns are not part of the final solution of the photogrammetric problem. Their introduction is, however, essential for the solution of the set of simultaneous equations. The reason is the following:

In general, the available data is redundant in that it gives more information that necessary to solve the photogrammetric problem. Were it nor for errors in the data, any part of it sufficient to solve the problem could be used and the rest discarded. We consider, therefore, that the true value A of a quantity is its measured value A_o plus an unknown error dA, so that

$$A = A_o + dA$$

In the case of random errors, the error term dA has associated with it a standard deviation J_A which expresses the probable accuracy of A_o. If the dA's were known, the problem would be solvable exactly, except for round-off. On the other hand, there is, in general, an infinite number of values for the dA's such that the simultaneous equations are satisfied. The problem is, therefore, to find that set of dA's which gives the most probable solution. This is done by the general method of least squares[1] which makes the weighted sum square of the dA's a minimum.

The residual unknowns introduced in the equations are the following:

a. dI_x - associated with the x photographic coordinate of an image point

b. dI_y - associated with the y photographic coordinate of an image point

c. $d\phi$ - associated with the known latitude of a ground point or an exposure station

d. $d\lambda$ - associated with the longitude of a ground point or an exposure station

e. dh - associated with the elevation of a ground point or an exposure station

E. NOTATION

1. Geometric Quantities

The following defines basic geometrical quantities of the photogrammetric system.

---

1. For more details on the method of least squares, see 1. and 2.

a. A letter with a bar over it represents a vector. The superscript o indicates a unit vector. Thus $\overline{V}^o$ is a unit vector along $\overline{V}$.

b. 0_1, 0_2, 0_3 represent 3 consecutive perspective centers.

c. $\overline{B_1} = \overline{0_2 0_1}$ = a vector joining 0_2 to 0_1 in the direction from 0_2 to 0_1.

d. $\overline{B_2} = \overline{0_3 0_2}$ = a vector joining 0_3 to 0_2 in the direction from 0_3 to 0_2.

e. $\overline{A}$ = vector joining an exposure station to a ground point.

f. $\overline{A_1}$, $\overline{A_2}$, $\overline{A_3}$ = vectors originating from 0_1, 0_2, 0_3 to the same ground point.

g. A_{1X}, A_{1Y}, A_{1Z} = geocentric components of $\overline{A}$.

h. A^o_{1X}, A^o_{1Y}, A^o_{1Z} or a^o_{1X}, a^o_{1Y}, or a^o_{1Z} are geocentric components of $\overline{A^o_1}$. They are also geocentric direction cosines of $\overline{A}_1$.

i. A^o_{1x}, A^o_{1y}, A^o_{1z} or a^o_{1x}, a^o_{1y}, or a^o_{1z} are photographic direction cosines of $\overline{A_1}$. They are also the photographic components of A^o_1.

j. $\overline{R}$ is the vector joining the geocentric origin to a perspective center. R_1 refers to $\overline{0}_1$.

2. Derived Quantities

The following expressions define certain derived quantities wherein a dot (·) or a cross (x) between two vectors represents the dot and cross products respectively of those vectors.

$$\overline{C}_{1B} = \overline{A}^o_2 \times \overline{A}^o_1$$

$$\overline{C}_{2B} = \overline{A}^o_3 \times \overline{A}^o_2$$

$$C_B = \text{SCALAR OF } \overline{C}_{1B}$$

$$C_{2B} = \text{SCALAR OF } \overline{C}_{2B}$$

$$g : \overline{B}_1 \cdot \overline{C}^o_{1B}$$

$$g_2 = \overline{B}_2 \cdot \overline{C}^{\circ}_{2B}$$

$$a_1 = \frac{\overline{B}_1}{C^2_{1B}} \cdot (\overline{A}^{\circ}_2 \times \overline{C}_{1B}) = \frac{\overline{C}_{1B}}{C^2_{1B}} \cdot (\overline{B}_1 \times \overline{A}^{\circ}_2)$$

$$a_2 = \frac{\overline{B}_1}{C^2_{1B}} \cdot (\overline{A}^{\circ}_1 \times \overline{C}_{1B}) = \frac{\overline{C}_{1B}}{C^2_{1B}} \cdot (\overline{B}_1 \times \overline{A}^{\circ}_1)$$

$$a'_2 = \frac{\overline{B}_2}{C^2_{2B}} \cdot (\overline{A}^{\circ}_3 \times \overline{C}_{2B}) = \frac{\overline{C}_{2B}}{C^2_{2B}} \cdot (\overline{B}_2 \times \overline{A}^{\circ}_3)$$

$$a'_3 = \frac{\overline{B}_2}{C^2_{2B}} \cdot (\overline{A}^{\circ}_2 \times \overline{C}_{2B}) = \frac{\overline{C}_{2B}}{C^2_{2B}} \cdot (\overline{B}_2 \times \overline{A}^{\circ}_2)$$

$$V_1 = a_1$$

$$V_2 = a_2$$

$$\overline{M}_1 = \frac{1}{C^2_{1B}} \left[g\, \overline{C}^{\circ}_{1B} + a_1 (\overline{C}_{1B} \times \overline{A}^{\circ}_2) \right]$$

$$\overline{N}_1 = \frac{1}{C^2_{1B}} \left[g\, (\overline{A}^{\circ}_2 \cdot \overline{A}^{\circ}_1)\, \overline{C}^{\circ}_{1B} + a_2 (C_{1B} \times \overline{A}^{\circ}_2) \right]$$

$$\overline{M}_2 = -\frac{1}{C^2_{1B}} \left[g\, (\overline{A}^{\circ}_2 \cdot \overline{A}^{\circ}_1)\, \overline{C}^{\circ}_{1B} + a_1 (\overline{C}_{1B} \times \overline{A}^{\circ}_1) \right]$$

$$\overline{N}_2 = \frac{1}{C^2_{1B}} \left[g\, \overline{C}^{\circ}_{1B} + a_2 (\overline{C}_{1B} \times \overline{A}^{\circ}_1) \right]$$

$$\overline{M}'_2 = \frac{1}{C^2_{2B}} \left[g_2 (\overline{A}^{\circ}_3 \cdot \overline{A}^{\circ}_2)\, \overline{C}^{\circ}_{2B} + a_3 (\overline{C}_{2B} \times \overline{A}^{\circ}_3) \right]$$

$$\overline{N}'_2 = -\frac{1}{C^2_{2B}} \left[g_2\, \overline{C}^{\circ}_{2B} + a'_2 (\overline{C}_{2B} \times \overline{A}^{\circ}_3) \right]$$

$$\overline{M}_3 = \frac{1}{C^2_{2B}} \left[g_2\, \overline{C}^{\circ}_{2B} + a_3 (\overline{C}_{2B} \times \overline{A}^{\circ}_2) \right]$$

$$\overline{N}_3 = -\frac{1}{C^2_{2B}} \left[g_2 (\overline{A}^{\circ}_3 \cdot \overline{A}^{\circ}_2)\, \overline{C}^{\circ}_{2B} + a'_2 (\overline{C}_{2B} \times \overline{A}^{\circ}_2) \right]$$

$$\overline{P} = \frac{1}{C^2_{1B}} \left[-(\overline{A}^{\circ}_1 \cdot \overline{A}^{\circ}_2)(B_1 \times \overline{A}^{\circ}_2) + (\overline{B}_1 \cdot \overline{A}^{\circ}_2 + 2a_1 \overline{A}^{\circ}_1 \cdot \overline{A}^{\circ}_2)\, \overline{C}_{1B} \right]$$

M_{ij} i = 1,2,3 j = 1,2,3 are the elements of the transformation matrix $[M]$ from photographic to geocentric axes

$\overline{I}$ vector joining a perspective center to the image I

I scalar of $\overline{I}$

t_{ij} i = 1,2,3 j = 1,2,3 are the elements of the matrix $[t]$ defined by:

$$[t] = \frac{[M]}{I}\left[[I] - [A^\circ][A^\circ]^T\right]$$

where

$$[A^\circ] = \begin{bmatrix} a_x^\circ \\ a_y^\circ \\ a_z^\circ \end{bmatrix}$$

$$[A^\circ]^T = \begin{bmatrix} a_x^\circ & a_y^\circ & a_z^\circ \end{bmatrix}$$

The superscript (') applied to the elements of a t matrix indicates that the matrix corresponds to the second photograph in a strip, while the superscript ('') indicates the third photograph in a strip, and the absence of a (') or ('') indicates the first photograph of a strip.

a semi-major axis of the geoid

e eccentricity of the geoid

3. Unknowns

a. Photographic Unknowns

These have been defined in paragraph D. The subscript 1,2, or 3 applied to dR_x, dR_y, or dR_z indicates that these quantities refer to the first, second, or third exposure station considered in a particular equation.

b. Residual Unknowns

Subscripts 1,2, or 3 added to dI_x and dI_y refer respectively to the images of the ground point under consideration

in the first, second, and third photograph considered

in a particular equation.

Further notation will be introduced as needed.

F. LISTING OF THE EQUATIONS

For the derivation of the coefficients of the exposure stations unknowns, see:

"Sequential Presentation of Analytical Aerotriangulation by the Revised Direct Geodetic Restraint Method"

ERDL;

and for the derivation of the coefficients of the residual unknowns, see:

"Analytical Aerial Trian gulation Error Analysis and Application of Compensating Equations to the General Block Triangulation and Adjustment Program"

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In the following, the condition enforced by every equation is briefly described. The equations being generally long, their listing will be in the following way: if, for example, an equation involves two successive camera stations and two image points, the unknowns will be:

dT_{1x}, dT_{1y}, dT_{1z}, dR_{1x}, dR_{1y}, dR_{1z}

dT_{2x}, dT_{2y}, dT_{2z}, dR_{2x}, dR_{2y}, dR_{2z}

dI_{1x}, dI_{1y}

dI_{2x}, dI_{2y}

The listings will give the coefficients of the above unknowns and the constant term as it appears on the right hand side of the equation.

I. EQUATION I - PASS POINT EQUATION

Two camera stations and the corresponding images of a ground point are involved.

The condition of intersection of the rays $\bar{A}_1$ and $\bar{A}_2$, from the perspective centers to the corresponding images, is expressed.

Coefficients

dT_{1X} : $v_1(\bar{A}_1^\circ \times \bar{C}_{1B}^\circ)_X$

dT_{1Y} : $v_1(\bar{A}_1^\circ \times \bar{C}_{1B}^\circ)_Y$

dT_{1Z} : $v_1(\bar{A}_1^\circ \times \bar{C}_{1B}^\circ)_Z$

dR_{1X} : C_{1BX}°

dR_{1Y} : C_{1BY}°

dR_{1Z} : C_{1BZ}°

dT_{2X} : $v_2(\bar{A}_2^\circ \times \bar{C}_{1B}^\circ)_X$

dT_{2Y} : $v_2(\bar{A}_2^\circ \times \bar{C}_{1B}^\circ)_Y$

dT_{2Z} : $v_2(\bar{A}_2^\circ \times \bar{C}_{1B}^\circ)_Z$

dR_{2X} : $-C_{1BX}^\circ$

dR_{2Y} : $-C_{1BY}^\circ$

dR_{2Z} : $-C_{1BZ}^\circ$

dI_{1X} : $v_1(C_{1BX}^\circ t_{11} + C_{1BY}^\circ t_{21} + C_{1BZ}^\circ t_{31})$

dI_{1Y} : $v_1(C_{1BX}^\circ t_{12} + C_{1BY}^\circ t_{22} + C_{1BZ}^\circ t_{32})$

dI_{2X} : $v_2(C_{1BX}^\circ t'_{11} + C_{1BY}^\circ t'_{21} + C_{1BZ}^\circ t'_{31})$

dI_{2Y} : $v_2(C_{1BX}^\circ t'_{12} + C_{1BY}^\circ t'_{22} + C_{1BZ}^\circ t'_{32})$

CONSTANT TERM = $(-B_{1X} C_{1BX}^\circ - B_{1Y} C_{1BY}^\circ - B_{1Z} C_{1BZ}^\circ)$

2. EQUATION 2 - DIFFERENTIAL SCALE RESTRAINT

Three camera stations and the corresponding images of a ground point are involved.

The condition of intersection at a point of the three rays $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, from the three perspective centers to the corresponding three image points, is expressed.

Coefficients

$$dT_{1X} : -\left\{\frac{(\bar{B}_1 \times \bar{A}_1^{\circ})_X (\bar{A}_1^{\circ} \cdot \bar{A}_2^{\circ}) - [-(\bar{B}_1 \cdot \bar{A}_1^{\circ}) + 2a_2 (\bar{A}_1^{\circ} \cdot \bar{A}_2^{\circ})]\, C_{1BX}}{(C_{1B})^2}\right\}$$

$$dT_{1Y} : -\left\{\frac{(\bar{B}_1 \times \bar{A}_1^{\circ})_Y (\bar{A}_1^{\circ} \cdot \bar{A}_2^{\circ}) - [-(\bar{B}_1 \cdot \bar{A}_1^{\circ}) + 2a_2 (\bar{A}_1^{\circ} \cdot \bar{A}_2^{\circ})]\, C_{1BY}}{(C_{1B})^2}\right\}$$

$$dT_{1Z} : -\left\{\frac{(\bar{B}_1 \times \bar{A}_1^{\circ})_Z (\bar{A}_1^{\circ} \cdot \bar{A}_2^{\circ}) - [-(\bar{B}_1 \cdot \bar{A}_1^{\circ}) + 2a_2 (\bar{A}_1^{\circ} \cdot \bar{A}_2^{\circ})]\, C_{1BZ}}{(C_{1B})^2}\right\}$$

$$dR_{1X} : -\frac{(\bar{A}_1^{\circ} \times \bar{C}_{1B})_X}{(C_{1B})^2}$$

$$dR_{1Y} : -\frac{(\bar{A}_1^{\circ} \times \bar{C}_{1B})_Y}{(C_{1B})^2}$$

$$dR_{1Z} : -\frac{(\bar{A}_1^{\circ} \times \bar{C}_{1B})_Z}{(C_{1B})^2}$$

$$dT_{2X} : -\left\{\frac{[-(\bar{B}_1 \cdot \bar{A}_1^{\circ}) + 2a_2 (\bar{A}_1^{\circ} \cdot \bar{A}_2^{\circ})]\, C_{1BX} - (\bar{B}_1 \times \bar{A}_2^{\circ})_X}{(C_{1B})^2}\right\}$$

$$+ \frac{(\bar{B}_2 \times \bar{A}_2^{\circ})_X - [(\bar{B}_2 \cdot \bar{A}_3^{\circ}) + 2a_2' (\bar{A}_2^{\circ} \cdot \bar{A}_3^{\circ})]\, C_{2BX}}{(C_{2B})^2}$$

$$dT_{2Y}:\ -\left\{\frac{[-(\bar{B}_1\cdot\bar{A}_1^{\circ})+2a_2\,(\bar{A}_1^{\circ}\cdot\bar{A}_2^{\circ})]\,C_{1BY}-(\bar{B}_1\times\bar{A}_2^{\circ})_Y}{(C_{1B})^2}\right\}$$

$$+\ \frac{(\bar{B}_2\times\bar{A}_2^{\circ})_Y-[(\bar{B}_2\cdot\bar{A}_3^{\circ})+2a_2'\,(\bar{A}_2^{\circ}\cdot\bar{A}_3^{\circ})]\,C_{2BY}}{(C_{2B})^2}$$

$$dT_{2Z}:\ -\left\{\frac{[-(\bar{B}_1\cdot\bar{A}_1^{\circ})+2a_2(\bar{A}_1^{\circ}\cdot\bar{A}_2^{\circ})]\,C_{1BZ}-(\bar{B}_1\times\bar{A}_2^{\circ})_Z}{(C_{1B})^2}\right\}$$

$$+\ \frac{(\bar{B}_2\times\bar{A}_2^{\circ})_Z-[(\bar{B}_2\cdot\bar{A}_3^{\circ})+2a_2'\,(\bar{A}_2^{\circ}\cdot\bar{A}_3^{\circ})]\,C_{2BZ}}{(C_{2B})^2}$$

$$dR_{2X}:\ \frac{(\bar{A}_3^{\circ}\times\bar{C}_{2B})_X}{(C_{2B})^2}+\frac{(\bar{A}_1^{\circ}\times\bar{C}_{1B})_X}{(C_{1B})^2}$$

$$dR_{2Y}:\ \frac{(\bar{A}_3^{\circ}\times\bar{C}_{2B})_Y}{(C_{2B})^2}+\frac{(\bar{A}_1^{\circ}\times\bar{C}_{1B})_Y}{(C_{1B})^2}$$

$$dR_{2Z}:\ \frac{(\bar{A}_3^{\circ}\times\bar{C}_{2B})_Z}{(C_{2B})^2}+\frac{(\bar{A}_1^{\circ}\times\bar{C}_{1B})_Z}{(C_{1B})^2}$$

$$dT_{3X}:\ \frac{[(\bar{B}_2\cdot\bar{A}_3^{\circ})+2a_2'\,(\bar{A}_2^{\circ}\cdot\bar{A}_3^{\circ})]\,C_{2BX}-(\bar{B}_2\times\bar{A}_3^{\circ})_X\,(\bar{A}_2^{\circ}\cdot\bar{A}_3^{\circ})}{(C_{2B})^2}$$

$$dT_{3Y}:\ \frac{[(\bar{B}_2\cdot\bar{A}_3^{\circ})+2a_2'\,(\bar{A}_2^{\circ}\cdot\bar{A}_3^{\circ})]\,C_{2BY}-(\bar{B}_2\times\bar{A}_3^{\circ})_Y\,(\bar{A}_2^{\circ}\cdot\bar{A}_3^{\circ})}{(C_{2B})^2}$$

$$dT_{3Z}:\ \frac{[(\bar{B}_2\cdot\bar{A}_3^{\circ})+2a_2'\,(\bar{A}_2^{\circ}\cdot\bar{A}_3^{\circ})]\,C_{2BZ}-(\bar{B}_2\times\bar{A}_3^{\circ})_Z\,(\bar{A}_2^{\circ}\cdot\bar{A}_3^{\circ})}{(C_{2B})^2}$$

$$dR_{3X}:\ -\frac{(\bar{A}_3^{\circ}\times\bar{C}_{2B})_X}{(C_{2B})^2}$$

$$dR_{3Y}: -\frac{(\bar{A}_3^{\circ} \times \bar{C}_{2B})_Y}{(C_{2B})^2}$$

$$dR_{3Z}: -\frac{(\bar{A}_3^{\circ} \times \bar{C}_{2B})_Z}{(C_{2B})^2}$$

$$dI_{1X}: -M_{2X}t_{11} - M_{2Y}t_{21} - M_{2Z}t_{31}$$

$$dI_{1Y}: -M_{2X}t_{12} - M_{2Y}t_{22} - M_{2Z}t_{32}$$

$$dI_{2X}: (N'_{2X} - N_{2X})t'_{11} + (N'_{2Y} - N_{2Y})t'_{21} + (N'_{2Z} - N_{2Z})t'_{31}$$

$$dI_{2Y}: (N'_{2X} - N_{2X})t'_{12} + (N'_{2Y} - N_{2Y})t'_{22} + (N'_{2Z} - N_{2Z})t'_{32}$$

$$dI_{3X}: M'_{2X}t''_{11} + M'_{2Y}t''_{21} + M'_{2Z}t''_{31}$$

$$dI_{3Y}: M'_{2X}t''_{12} + M'_{2Y}t''_{22} + M'_{2Z}t''_{32}$$

$$\text{CONSTANT TERM} = -B_{1X}Q_X - B_{1Y}Q_Y - B_{1Z}Q_Z - [B_{2X}L'_X + B_{2Y}L'_Y + B_{2Z}L'_Z]$$

3. EQUATIONS 3 and 4 - COMPLETE GROUND CONTROL EQUATIONS

One camera station and the corresponding image of a ground point are involved, where the latitude ϕ, longitude λ, and elevation h are known.

Definitions

$\bar{A}_{01}$ is a vector to be defined as follows:

Associate a component of $\bar{A}_{01}$ equal to 0.7 with the coordinate that has the smallest $\bar{A}$ component in absolute value. Associate a component of $\bar{A}_{01}$ equal to 0.0 with the coordinate that has the intermediate-sized $\bar{A}$ component in absolute value and associate a component of $\bar{A}_{01}$ equal to $-0.7 \frac{A_{MIN}}{A_{MAX}}$ with the coordinate that has the largest $\bar{A}$ component in absolute value. A_{MIN} and A_{MAX} are such that their absolute values are the smallest and the largest respectively of the absolute values of the components of $\bar{A}$. $\bar{A}_{01}$ is perpendicular to $\bar{A}$. $\bar{A}^{\bullet}_{01}$ is obtained by normalizing $\bar{A}_{01}$. $\bar{A}^{\bullet}_{02} = \bar{A}^{\bullet}_{n} \times \bar{A}^{\bullet}_{01}$.

Equation 3 expresses the condition of intersection of the ray $\bar{A}$ with the vector $\bar{A}^{\bullet}_{01}$ passing through the ground point. Equation 4 expresses the condition of intersection of the ray $\bar{A}$ with the vector $\bar{A}^{\bullet}_{02}$ passing through the ground point.

Equations 3 and 4 condisered together express the condition that the ray A passes through the ground point.

a. Equation 3 Coefficients

dT_X : $-V_{2C} A^{\bullet}_{02X}$

dT_Y : $-V_{2C} A^{\bullet}_{02Y}$

dT_Z : $-V_{2C}\, A^{\circ}_{O2Z}$

dR_X : $A^{\circ}_{nZ}\, A^{\circ}_{O1Y}$

dR_Y : $-A^{\circ}_{nZ}\, A^{\circ}_{O1X}$

dR_Z : $-(A^{\circ}_{nX} A^{\circ}_{O1Y} - A^{\circ}_{nY} A^{\circ}_{O1X})$

dI_X : $V_{2C}\,(A^{\circ}_{O2X} t_{11} + A^{\circ}_{O2Y} t_{21} + A^{\circ}_{O2Z} t_{31})$

dI_Y : $V_{2C}\,(A^{\circ}_{O2X} t_{12} + A^{\circ}_{O2Y} t_{22} + A^{\circ}_{O2Z} t_{32})$

$$d\phi : \quad -A^{\circ}_{O2X}\left[\frac{a(1-e^2)\sin\phi\cos\lambda}{(1-e^2\sin^2\phi)^{3/2}} + h\sin\phi\cos\lambda\right]$$

$$-A^{\circ}_{O2Y}\left[\frac{a(1-e^2)\sin\phi\sin\lambda}{(1-e^2\sin^2\phi)^{3/2}} + h\sin\phi\sin\lambda\right]$$

$$+A^{\circ}_{O2Z}\left[\frac{a(1-e^2)\cos\phi}{(1-e^2\sin^2\phi)^{3/2}} + h\cos\phi\right]$$

$$d\lambda : \quad -A^{\circ}_{O2X}\left[\frac{a\cos\phi\sin\lambda}{\sqrt{1-e^2\sin^2\phi}} + h\cos\phi\sin\lambda\right]$$

$$+A^{\circ}_{O2Y}\left[\frac{a\cos\phi\cos\lambda}{\sqrt{1-e^2\sin^2\phi}} + h\cos\phi\cos\lambda\right]$$

dh : $A^{\circ}_{O2X}\cos\phi\cos\lambda + A^{\circ}_{O2Y}\cos\phi\sin\lambda + A^{\circ}_{O2Z}\sin\phi$

CONSTANT TERM = $[-D_X A^{\circ}_{O2X} - D_Y A^{\circ}_{O2Y} - D_Z A^{\circ}_{O2Z}]$

b. Equation 4 Coefficients

$$dT_X \quad : \quad -v_{2c}\, A^\circ_{02X}$$

$$dT_Y \quad : \quad -v_{2c}\, A^\circ_{02Y}$$

$$dT_Z \quad : \quad -v_{2c}\, A^\circ_{02Z}$$

$$dR_X \quad : \quad -\frac{A^\circ_{nY}}{\sqrt{(A^\circ_{nX})^2 + (A^\circ_{nY})^2}}$$

$$dR_Y \quad : \quad \frac{A^\circ_{nX}}{\sqrt{(A^\circ_{nX})^2 + (A^\circ_{nY})^2}}$$

$$dR_Z \quad : \quad 0$$

$$dI_X \quad : \quad -v_{2c}\,(A^\circ_{01X}\, t_{11} + A^\circ_{01Y}\, t_{21} + A^\circ_{01Z}\, t_{31})$$

$$dI_Y \quad : \quad -v_{2c}\,(A^\circ_{01X}\, t_{12} + A^\circ_{01Y}\, t_{22} + A^\circ_{01Z}\, t_{32})$$

$$d\phi \quad : \quad A^\circ_{01X}\left[\frac{a(1-e^2)\sin\phi\cos\lambda}{(1-e^2\sin^2\phi)^{3/2}} + h\sin\phi\cos\lambda\right]$$

$$+ A^\circ_{01Y}\left[\frac{a(1-e^2)\sin\phi\sin\lambda}{(1-e^2\sin^2\phi)^{3/2}} + h\sin\phi\sin\lambda\right]$$

$$- A^\circ_{01Z}\left[\frac{a(1-e^2)\cos\phi}{(1-e^2\sin^2\phi)^{3/2}} + h\cos\phi\right]$$

$$d\lambda \quad : \quad A^\circ_{01X}\left[\frac{a\cos\phi\sin\lambda}{\sqrt{1-e^2\sin^2\phi}} + h\cos\phi\sin\lambda\right]$$

$$- A^\circ_{01Y}\left[\frac{a\cos\phi\cos\lambda}{\sqrt{1-e^2\sin^2\phi}} + h\cos\phi\cos\lambda\right]$$

$$dh : -A^{\bullet}_{01X} \cos\phi \cos\lambda - A^{\bullet}_{01Y} \cos\phi \sin\lambda - A^{\bullet}_{01Z} \sin\phi$$

$$\text{CONSTANT TERM} = D_X A^{\bullet}_{01X} + D_Y A^{\bullet}_{01Y}$$

4. EQUATION 5 - HORIZONTAL GROUND CONTROL POINT

Two camera stations and the corresponding ground point images are involved where the longitude λ is known.

The condition that the common perpendicular between rays $\bar{A}_1$ and $\bar{A}_2$ intersecting ray $\bar{A}_1$ lies in the plane of equal longitude, is expressed.

Auxiliary Definitions

$\alpha = \sin \lambda$

$\beta = \cos \lambda$

Coefficients

dT_{1X} : $[\alpha A^{\circ}_{1X} G_X - \beta (A^{\circ}_{1Y} G_X - a_1 A^{\circ}_{1Z})]$

dT_{1Y} : $[\alpha (A^{\circ}_{1X} G_Y + a_1 A^{\circ}_{1Z}) - \beta A^{\circ}_{1Y} G_Y]$

dT_{1Z} : $[\alpha (A^{\circ}_{1X} G_Z - a_1 A^{\circ}_{1Y}) - \beta (A^{\circ}_{1Y} G_Z + a_1 A^{\circ}_{1X})]$

dR_{1X} : $[(\alpha A^{\circ}_{1X} - \beta A^{\circ}_{1Y}) L_X + \alpha]$

dR_{1Y} : $[(\alpha A^{\circ}_{1X} - \beta A^{\circ}_{1Y}) L_Y - \beta]$

dR_{1Z} : $[(\alpha A^{\circ}_{1X} - \beta A^{\circ}_{1Y}) L_Z]$

dT_{2X} : $[(\alpha A^{\circ}_{1X} - \beta A^{\circ}_{1Y}) P_X]$

dT_{2Y} : $[(\alpha A^{\circ}_{1X} - \beta A^{\circ}_{1Y}) P_Y]$

dT_{2Z} : $[(\alpha A^{\circ}_{1X} - \beta A^{\circ}_{1Y}) P_Z]$

dR_{2X} : $[(\alpha A^{\circ}_{1X} - \beta A^{\circ}_{1Y}) (-L_X)]$

dR_{2Y} : $[(\alpha A^{\circ}_{1X} - \beta A^{\circ}_{1Y})(-L_Y)]$

dR_{2Z} : $[(\alpha A^{\circ}_{1X} - \beta A^{\circ}_{1Y})(-L_Z)]$

dI_{1X} : $a_1(\alpha t_{11} - \beta t_{21}) + (\alpha A^{\circ}_{1X} - \beta A^{\circ}_{1Y})(M_{1X} t_{11} + M_{1Y} t_{21} + M_{1Z} t_{31})$

dI_{1Y} : $a_1(\alpha t_{12} - \beta t_{22}) + (\alpha A^{\circ}_{1X} - \beta A^{\circ}_{1Y})(M_{1X} t_{12} + M_{1Y} t_{22} + M_{1Z} t_{32})$

dI_{2X} : $(\alpha A^{\circ}_{1X} - \beta A^{\circ}_{1Y})(N_{1X} t'_{11} + N_{1Y} t'_{21} + N_{1Z} t'_{31})$

dI_{2Y} : $(\alpha A^{\circ}_{1X} - \beta A^{\circ}_{1Y})(N_{1X} t'_{12} + N_{1Y} t'_{22} + N_{1Z} t'_{32})$

$d\lambda$: $\beta(R_{1X} + a_1 A^{\circ}_{1X}) + \alpha(R_{1Y} + a_1 A^{\circ}_{1Y})$

CONSTANT TERM : $\beta(R_{1Y} + A_{1Y}) - \alpha(R_{1X} + A_{1X})$

5. EQUATION 6 - HORIZONTAL GROUND CONTROL POINT

Two camera stations and the corresponding images of a ground point are involved where the longitude λ and latitude ϕ are known.

The condition that the point of intersection with the ray $\bar{A}_1$, of the common perpendicular to rays $\bar{A}_1$ and A_2, lies in the prime vertical plane cooresponding to ϕ and λ. The prime vertical plane being the plane containing the normal to the geoid at the ground point and tangent to the circle of latitude equal to ϕ passing through the ground point.

Auxiliary Definitions

$$\gamma = \cos \lambda \sin \phi$$

$$\delta = \sin \lambda \sin \phi$$

$$\epsilon = - \cos \phi$$

$$\zeta = \frac{ae^2 \sin\phi \cos\phi}{\sqrt{1-e^2 \sin^2 \phi}}$$

Coefficients

$$dT_{1x} \quad : \quad [\gamma A^{\circ}_{1x} G_x + \varepsilon (A^{\circ}_{1y} G_x - a_1 A^{\circ}_{1z}) + \epsilon (A^{\circ}_{1z} G_x + a_1 A^{\circ}_{1y})]$$

$$dT_{1y} \quad : \quad [\gamma (A^{\circ}_{1x} G_y + a_1 A^{\circ}_{1z}) + \delta A^{\circ}_{1y} G_y + \epsilon (A^{\circ}_{1z} G_y - a_1 A^{\circ}_{1x})]$$

$$dT_{1z} \quad : \quad [\gamma (A^{\circ}_{1x} G_z - a_1 A^{\circ}_{1y}) + \delta (A^{\circ}_{1y} G_z + a_1 A^{\circ}_{1x}) + \epsilon A^{\circ}_{1z} G_z]$$

$$dR_{1x} \quad : \quad [(\gamma A^{\circ}_{1x} + \delta A^{\circ}_{1y} + \epsilon A^{\circ}_{1z}) L_x + \gamma]$$

$$dR_{1y} \quad : \quad [(\gamma A^{\circ}_{1x} + \delta A^{\circ}_{1y} + \epsilon A^{\circ}_{1z}) L_y + \delta]$$

$$dR_{1z} \quad : \quad [(\gamma A^{\circ}_{1x} + \delta A^{\circ}_{1y} + \epsilon A^{\circ}_{z}) L_z + \epsilon]$$

dT_{2X} : $[(\gamma A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \epsilon A^{\circ}_{1Z}) P_X]$

dT_{2Y} : $[(\gamma A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \epsilon A^{\circ}_{1Z}) P_Y]$

dT_{2Z} : $[(\gamma A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \epsilon A^{\circ}_{1Z}) P_Z]$

dR_{2X} : $[(\gamma A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \epsilon A^{\circ}_{1Z}) (-L_X)]$

dR_{2Y} : $[(\gamma A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \epsilon A^{\circ}_{1Z}) (-L_Y)]$

dR_{2Z} : $[(\gamma A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \epsilon A^{\circ}_{1Z}) (-L_Z)]$

dI_{1X} : $a_1(\gamma t_{11} + \delta t_{21} + \epsilon t_{31}) + (\gamma A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \epsilon A^{\circ}_{1Z})(M_{1X} t_{11} + M_{1Y} t_{21} + M_{1Z} t_{31})$

dI_{1Y} : $a_1(\gamma t_{12} + \delta t_{22} + \epsilon t_{32}) + (\gamma A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \epsilon A^{\circ}_{1Z})(M_{1X} t_{12} + M_{1Y} t_{22} + M_{1Z} t_{32})$

dI_{2X} : $(\gamma A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \epsilon A^{\circ}_{1Z})(N_{1X} t'_{11} + N_{1Y} t'_{21} + N_{1Z} t'_{31})$

dI_{2Y} : $(\gamma A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \epsilon A^{\circ}_{1Z})(N_{1X} t'_{12} + N_{1Y} t'_{22} + N_{1Z} t'_{32})$

$d\phi$: $\cos\lambda\cos\phi\,(R_{1X} + a_1 A^{\circ}_{1X}) + \sin\lambda\cos\phi\,(R_{1Y} + a_1 A^{\circ}_{1Y})$

$$+ \sin\phi\,(R_{1Z} + a_1 A^{\circ}_{1Z}) - ae^2 \frac{\cos^2\phi - \sin^2\phi(1 - e^2\sin^2\phi)}{(1 - e^2\sin^2\phi)^{3/2}}$$

$d\lambda$: $-\sin\lambda\sin\phi\,(R_{1X} + a_1 A^{\circ}_{1X}) + \cos\lambda\sin\phi\,(R_{1Y} + a_1 A^{\circ}_{1Y})$

CONSTANT TERM $= 3 - [\gamma(R_{1X} + A_{1X}) + \delta(R_{1Y} + A_{1Y}) + \epsilon(R_{1Z} + A_{1Z})]$

6. EQUATION 6A - HORIZONTAL GROUND POINT

It involves two camera stations and the corresponding two images of a ground point whose latitude ϕ is known.

It expresses the condition the the point of intersection of the ray $\bar{A}_1$, with the common perpendicular to rays $\bar{A}_1$ and $\bar{A}_2$, lies on the cone of equal latitude ϕ.

Auxiliary Definations

$$t = \frac{a e^2 \sin \phi}{\sqrt{1 - e^2 \sin^2 \phi}}$$

$$\Psi = 2(R_{1X} + A_{1X}) = 2(R_{1X} + a_1 A^*_{1X})$$

$$\delta = 2(R_{1Y} + A_{1Y}) = 2(R_{1Y} + a_1 A^*_{1Y})$$

$$\omega = -\frac{2(R_{1Z} + A_{1Z} + t)}{\tan^2 \phi} = -\frac{2(R_{1Z} + a_1 A^*_{1Z} + t)}{\tan^2 \phi}$$

$$\Omega = \psi A^*_{1X} + \delta A^*_{1Y} + \omega A^*_{1Z}$$

Coefficients

dT_{1X} : $[\psi A^*_{1X} G_X + \delta(A^*_{1Y} G_X - a_1 A^*_{1Z}) + \omega(A^*_{1Z} G_X + a_1 A^*_{1Y})]$

dT_{1Y} : $[\psi(A^*_{1X} G_Y + a_1 A^*_{1Z}) + \delta A^*_{1Y} G_Y + \omega(A^*_{1Z} G_Y - a_1 A^*_{1X})]$

dT_{1Z} : $[\psi(A^*_{1X} G_Z - a_1 A^*_{1Y}) + \delta(A^*_{1Y} G_Z + a_1 A^*_{1X}) + \omega A^*_{1Z} G_Z]$

dR_{1X} : $[(\psi A^*_{1X} + \delta A^*_{1Y} + \omega A^*_{1Z}) L_X + \psi]$

dR_{1Y} : $[(\psi A^*_{1X} + \delta A^*_{1Y} + \omega A^*_{1Z}) L_Y + \delta]$

dR_{1Z} : $[(\psi A^*_{1X} + \delta A^*_{1Y} + \omega A^*_{1Z}) L_Z + \omega]$

dT_{2X} : $[(\psi A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \omega A^{\circ}_{1Z}) P_X]$

dT_{2Y} : $[(\psi A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \omega A^{\circ}_{1Z}) P_Y]$

dT_{2Z} : $[(\psi A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \omega A^{\circ}_{1Z}) P_Z]$

dR_{2X} : $[(\psi A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \omega A^{\circ}_{1Z}) (-L_X)]$

dR_{2Y} : $[(\psi A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \omega A^{\circ}_{1Z}) (-L_Y)]$

dR_{2Z} : $[(\psi A^{\circ}_{1X} + \delta A^{\circ}_{1Y} + \omega A^{\circ}_{1Z}) (-L_Z)]$

dI_{1X} : $a_1(\psi t_{11} + \delta t_{21} + \omega t_{31}) + \Omega(M_{1X} t_{11} + M_{1Y} t_{21} + M_{1Z} t_{31})$

dI_{1Y} : $a_1(\psi t_{12} + \delta t_{22} + \omega t_{32}) + \Omega(M_{1X} t_{12} + M_{1Y} t_{22} + M_{1Z} t_{32})$

dI_{2X} : $\Omega(N_{1X} t'_{11} + N_{1Y} t'_{21} + N_{1Z} t'_{31})$

dI_{2Y} : $\Omega(N_{1X} t'_{12} + N_{1Y} t'_{22} + N_{1Z} t'_{32})$

$d\phi$: $\frac{\omega}{2} \text{TAN}\, \phi \,(1 + \text{TAN}^2 \phi) + \frac{ae^2 \cos \phi}{(1 - e^2 \text{SIN}^2 \phi)^{3/2}} (\omega)$

$$\text{CONSTANT TERM} = \frac{(R_{1Z} + A_{1Z} + t)^2}{\text{TAN}^2 \phi} - (R_{1X} + A_{1X})^2 - (R_{1Y} + A_{1Y})^2$$

7. EQUATION 7 - VERTICAL GROUND CONTROL POINT

Two camera stations and the corresponding images of a ground point are involved, where the elevation h is known.

The condition that the point H , intersection of ray $\bar{A}_1$ with the common perpendicular to rays $\bar{A}_1$ and $\bar{A}_2$, lies on a sphere centered on the Z geoentric axes and having the elevation h along the circle of latitude ϕ . If the latitude ϕ of the ground point is not given it is taken as that of point H .

Auxiliary Definitions

$$n = \frac{a}{\sqrt{1-e^2 \sin^2 \phi}}$$

$$\eta = \frac{R_{1X} + A_{1X}}{n + h}$$

$$\theta = \frac{R_{1Y} + A_{1Y}}{n + h}$$

$$\zeta = \frac{R_{1Z} + A_{1Z}}{n + h}$$

$$t = \frac{a e^2 \sin \phi}{\sqrt{1-e^2 \sin^2 \phi}}$$

Coefficients

$$dT_{1X} : \quad [(\eta A^{\bullet}_{1X} G_X) + \theta(A^{\bullet}_{1Y} G_X - a_1 A^{\bullet}_{1Z}) + \zeta(A^{\bullet}_{1Z} G_X + a_1 A^{\bullet}_{1Y})]$$

$$dT_{1Y} : \quad [\eta(A^{\bullet}_{1X} G_Y + a_1 A^{\bullet}_{1Z}) + \theta A^{\bullet}_{1Y} G_Y + \zeta(A^{\bullet}_{1Z} G_Y - a_1 A^{\bullet}_{1X})]$$

$$dT_{1Z} : \quad \{\eta(A^{\bullet}_{1X} G_Z - a_1 A^{\bullet}_{1Y}) + \theta(A^{\bullet}_{1Y} G_Z + a_1 A^{\bullet}_{1X}) + \zeta A^{\bullet}_{1Z} G_Z]$$

dR_{1X} : $[(\xi A_{1X}^{\circ} + \theta A_{1Y}^{\circ} + \zeta A_{1Z}^{\circ}) L_X + \xi]$

dR_{1Y} : $[(\xi A_{1X}^{\circ} + \theta A_{1Y}^{\circ} + \zeta A_{1Z}^{\circ}) L_Y + \theta]$

dR_{1Z} : $[(\xi A_{1X}^{\circ} + \theta A_{1Y}^{\circ} + \zeta A_{1Z}^{\circ}) L_Z + \zeta]$

dT_{2X} : $[(\xi A_{1X}^{\circ} + \theta A_{1Y}^{\circ} + \zeta A_{1Z}^{\circ}) P_X]$

dT_{2Y} : $[(\xi A_{1X}^{\circ} + \theta A_{1Y}^{\circ} + \zeta A_{1Z}^{\circ}) P_Y]$

dT_{2Z} : $[(\xi A_{1X}^{\circ} + \theta A_{1Y}^{\circ} + \zeta A_{1Z}^{\circ}) P_Z]$

dR_{2X} : $[(\xi A_{1X}^{\circ} + \theta A_{1Y}^{\circ} + \zeta A_{1Z}^{\circ})(-L_X)]$

dR_{2Y} : $[(\xi A_{1X}^{\circ} + \theta A_{1Y}^{\circ} + \zeta A_{1Z}^{\circ})(-L_Y)]$

dR_{2Z} : $[(\xi A_{1X}^{\circ} + \theta A_{1Y}^{\circ} + \zeta A_{1Z}^{\circ})(-L_Z)]$

dI_{1X} : $a_1 (\eta t_{11} + \theta t_{21} + \zeta t_{31}) + (\eta A_{1X}^{\circ} + \theta A_{1Y}^{\circ} + \zeta A_{1Z}^{\circ})$
$(M_{1X} t_{11} + M_{1Y} t_{21} + M_{1Z} t_{31})$

dI_{1Y} : $a_1 (\eta t_{12} + \theta t_{22} + \zeta t_{32}) + (\eta A_{1X}^{\circ} + \theta A_{1Y}^{\circ} + \zeta A_{1Z}^{\circ})$
$(M_{1X} t_{12} + M_{1Y} t_{22} + M_{1Z} t_{32})$

dI_{2X} : $(\eta A_{1X}^{\circ} + \theta A_{1Y}^{\circ} + \zeta A_{1Z}^{\circ})(N_{1X} t'_{11} + N_{1Y} t'_{21} + N_{1Z} t'_{31})$

dI_{2Y} : $(\eta A_{1X}^{\circ} + \theta A_{1Y}^{\circ} + \zeta A_{1Z}^{\circ})(N_{1X} t'_{12} + N_{1Y} t'_{22} + N_{1Z} t'_{32})$

dh : -1

CONSTANT TERM $= \frac{1}{2} \left\{ n + h - \frac{[(R_{1X}+A_{1X})^2 + (R_{1Y}+A_{1Y})^2 + (R_{1Z}+A_{1Z})^2]}{n+h} \right\}$

8. EQUATION 56 - HORIZONTAL GROUND POINT APPEARING ON ONE PHOTOGRAPH

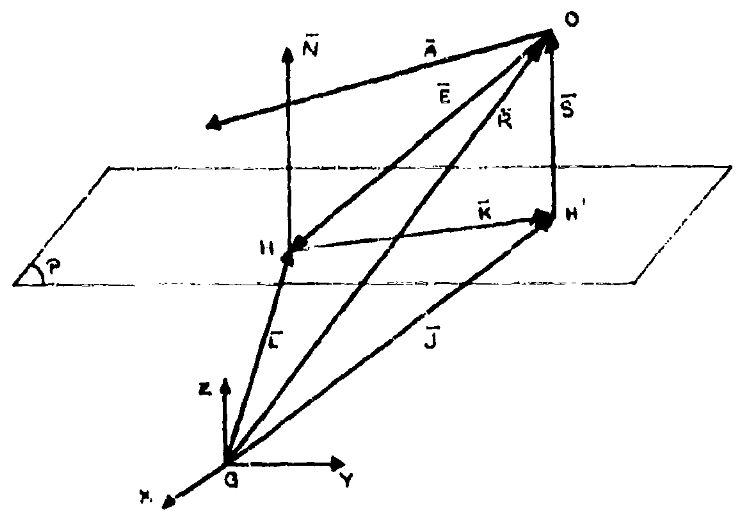
One camera station and the corresponding image of a ground point are involved where the longitude λ and the latitude ϕ of the ground point are known.

The condition that the ray $\bar{A}$ intersects the normal $\bar{N}$ to the geoid at the point of latitude ϕ and longitu..e λ is expressed.

Contrary to the other equations the derivation of <u>all</u> the coefficients of equation 56 is to be found in (   ).

Derived Quantities and Auxilary Definitions

Some of the vectors used can best be defined by the sketch shown below:



O = exposure station

N = Vector normal to the geoid at the point of latitude ϕ and longitude λ so that

$$N_x^{\circ} = \cos\phi \cos\lambda$$

$$N_y^{\circ} = \cos\phi \sin\lambda$$

$$N_z^{\circ} = \sin\phi$$

H = Arbitrary point on $\overline{N}$ taken usually at an altitude g equal to the estimated lowest altitude of the photographed area. g can be taken as zero.

P = Plane perpendicular to $\overline{N}$ passing through H

H′= Orthogonal projection of O on P

GX, GY, GZ = Geocentric axes

$\overline{L} = \overrightarrow{GH}$

$$L_x = \frac{a N^\circ_x}{\sqrt{1-e^2 \sin^2 \phi}} + g N^\circ_x$$

$$L_y = \frac{a N^\circ_y}{\sqrt{1-e^2 \sin^2 \phi}} + g N^\circ_y$$

$$L_z = \frac{a(1-e^2) N^\circ_z}{\sqrt{1-e^2 \sin^2 \phi}} + g N^\circ_z$$

$$\overline{R} = \overrightarrow{GO}$$

$$\overline{E} = \overrightarrow{OH} = \overline{L} - \overline{R}$$

$$\overline{K} = \overrightarrow{HH'} = -\overline{E} + (\overline{E}, \overline{N}^\circ)\ \overline{N}^\circ$$

$$\overline{S} = \overrightarrow{H'O} = -(\overline{E}, \overline{N}^\circ)\ \overline{N}^\circ$$

$$\overline{J} = \overrightarrow{GH'} = \overline{R} - \overline{S}$$

$$\overline{C}_E = \overline{A}^\circ \times \overline{K}^\circ$$

$$V_2 = \frac{\overline{C}_E}{C_E^2} \cdot (\overline{K}^\circ \times \overline{E})$$

$$V_1 = \frac{\overline{C}_E}{C_E^2} \cdot (\overline{E} \times \overline{A}^\circ)$$

$$\overline{W} = \left(\frac{V_1}{K} - 1\right) \overline{C}^\circ_E - \frac{V_1}{K} \left(\overline{C}^\circ_E \cdot \overline{N}^\circ\right) \overline{N}^\circ$$

$$\overline{W}' = \frac{V_1}{K} \left[(\overline{E} \cdot \overline{N}^\circ) \overline{C}^\circ_E + (\overline{C}^\circ_E \cdot \overline{N}^\circ) \overline{E} \right]$$

$$\delta' = \cos\phi \cos\lambda$$

$$\alpha' = -\sin\phi \cos\lambda$$

$$\beta' = -\cos\phi \sin\lambda$$

$$\gamma' = -\sin\phi \sin\lambda$$

$$\epsilon = \cos\phi$$

$$\alpha = \frac{a(1-e^2)\alpha'}{(1-e^2\sin^2\phi)^{3/2}} + g\alpha'$$

$$\beta = \frac{a\beta'}{\sqrt{(1-e^2\sin^2\phi)}} + g\beta'$$

$$\gamma = \frac{a(1-e^2)\gamma'}{(1-e^2\sin^2\phi)^{3/2}} + g\gamma'$$

$$\delta = \frac{a\delta}{\sqrt{(1-e^2\sin^2\phi)}} + g\delta'$$

$$\varepsilon = \frac{a(1-e^2)\varepsilon'}{(1-e^2\sin^2\phi)^{3/2}} + g\varepsilon'$$

Coefficients

dT_x : $V_e(\bar{A}_n^{\bullet} \times \bar{C}_E^{\bullet})_x$

dT_Y : $V_e(\bar{A}_n^{\bullet} \times \bar{C}_E^{\bullet})_Y$

dT_z : $V_e(\bar{A}_n^{\bullet} \times \bar{C}_E^{\bullet})_z$

dR_x : W_x

dR_Y : W_Y

$dR_Z \quad : \quad W_Z$

$dI_X \quad : \quad V_Z\,(\dot{C}_{EX}\, t_{11} + \dot{C}_{EY}\, t_{21} + \dot{C}_{EZ}\, t_{31})$

$dI_Y \quad : \quad V_Z\,(\dot{C}_{EX}\, t_{12} + \dot{C}_{EY}\, t_{22} + \dot{C}_{EZ}\, t_{32})$

$d\phi \quad : \quad -\alpha W_X - \gamma W_Y - \epsilon W_Z + \alpha' W'_X + \gamma' W'_Y + \epsilon' W'_Z$

$d\lambda \quad : \quad -\beta W_X - \delta W_Y + \beta' W'_X + \delta' W'_Y$

CONSTANT TERM $= -(\bar{E} \cdot \bar{\dot{C}}_E)$

9. EQUATION 70 - EQUI-ELEVATION GROUND POINT

Two exposure stations and the corresponding images of two ground points having the same elevations are involved, each ground point having an image in each photograph.

Superscripts (1) and (2) will refer to the 1st and 2nd ground points respectively. It expresses the condition that the point $H^{(1)}$ intersection of ray $\bar{A}_1^{(1)}$ with the common perpendicular to rays $\bar{A}_1^{(1)}$ and $\bar{A}_2^{(1)}$ has an elevation equal to point $H^{(2)}$ defined similarly to $H^{(1)}$ but for the second ground point.

Auxiliary Definitions

$$K_x = 2\,(A_{1x}^{(1)} - A_{1x}^{(2)})$$

$$K_y = 2\,(A_{1y}^{(1)} - A_{1y}^{(2)})$$

$$K_z = \frac{2}{1-e^2}\,(A_{1z}^{(1)} - A_{1z}^{(2)})$$

$$\tau_x = 2\,(R_{1x} + A_{1x}^{(1)})$$

$$\tau_y = 2\,(R_{1y} + A_{1y}^{(1)})$$

$$\tau_z = \frac{2}{1-e^2}\,(R_{1z} + A_{1z}^{(1)})$$

$$V_x = -2\,(R_{1x} + A_{1x}^{(2)})$$

$$V_y = -2\,(R_{1y} + A_{1y}^{(2)})$$

$$V_z = -\frac{2}{1-e^2}\,(R_{1z} + A_{1z}^{(2)})$$

<u>Coefficients</u>

$$dT_{1x} \;:\; \left\{ \left[A^{\circ}_{1x} G_x \tau_x + (A^{\circ}_{1y} G_x - A_{1z}) \tau_y + (A^{\circ}_{1z} G_x + A_{1y}) \tau_z \right]^{(1)} + \left[A^{\circ}_{1x} G_x \nu_x + (A^{\circ}_{1y} G_x - A_{1z}) \nu_y + (A^{\circ}_{1z} G_x + A_{1y}) \nu_z \right]^{(2)} \right\}$$

$$dT_{1y} \;:\; \left\{ \left[(A^{\circ}_{1x} G_y + A_{1z}) \tau_x + A^{\circ}_{1y} G_y \tau_y + (A^{\circ}_{1z} G_y - A_{1x}) \tau_z \right]^{(1)} + \left[(A^{\circ}_{1x} G_y + A_{1z}) \nu_x + A^{\circ}_{1y} G_y \nu_y + (A^{\circ}_{1z} G_y - A_{1x}) \nu_z \right]^{(2)} \right\}$$

$$dT_{1z} \;:\; \left\{ \left[(A^{\circ}_{1x} G_z - A_{1y}) \tau_x + (A^{\circ}_{1y} G_z + A_{1x}) \tau_y + A^{\circ}_{1z} G_z \tau_z \right]^{(1)} + \left[(A^{\circ}_{1x} G_z - A_{1y}) \nu_x + (A^{\circ}_{1y} G_z + A_{1x}) \nu_y + A^{\circ}_{1z} G_z \nu_z \right]^{(2)} \right\}$$

$$dR_{1x} \;:\; \left\{ \left[(A^{\circ}_{1x} \tau_x + A^{\circ}_{1y} \tau_y + A^{\circ}_{1z} \tau_z) L_x \right]^{(1)} + \left[(A^{\circ}_{1x} \nu_x + A^{\circ}_{1y} \nu_y + A^{\circ}_{1z} \nu_z) L_x \right]^{(2)} + K_x \right\}$$

$$dR_{1y} \;:\; \left\{ \left[(A^{\circ}_{1x} \tau_x + A^{\circ}_{1y} \tau_y + A^{\circ}_{1z} \tau_z) L_y \right]^{(1)} + \left[(A^{\circ}_{1x} \nu_x + A^{\circ}_{1y} \nu_y + A^{\circ}_{1z} \nu_z) L_y \right]^{(2)} + K_y \right\}$$

$$dR_{1z} \;:\; \left\{ \left[(A^{\circ}_{1x} \tau_x + A^{\circ}_{1y} \tau_y + A^{\circ}_{1z} \tau_z) L_z \right]^{(1)} + \left[(A^{\circ}_{1x} \nu_x + A^{\circ}_{1y} \nu_y + A^{\circ}_{1z} \nu_z) L_z \right]^{(2)} + K_z \right\}$$

$$dT_{2x} \;:\; \left\{ \left[(A^{\circ}_{1x} \tau_x + A^{\circ}_{1y} \tau_y + A^{\circ}_{1z} \tau_z) \mathcal{P}_x \right]^{(1)} + \left[(A^{\circ}_{1x} \nu_x + A^{\circ}_{1y} \nu_y + A^{\circ}_{1z} \nu_z) \mathcal{P}_x \right]^{(2)} \right\}$$

$$dT_{2y} \;:\; \left\{ \left[(A^{\circ}_{1x} \tau_x + A^{\circ}_{1y} \tau_y + A^{\circ}_{1z} \tau_z) \mathcal{P}_y \right]^{(1)} + \left[(A^{\circ}_{1x} \nu_x + A^{\circ}_{1y} \nu_y + A^{\circ}_{1z} \nu_z) \mathcal{P}_y \right]^{(2)} \right\}$$

$$dT_{2z} \;:\; \left\{ \left[(A^{\circ}_{1x} \tau_x + A^{\circ}_{1y} \tau_y + A^{\circ}_{1z} \tau_z) \mathcal{P}_z \right]^{(1)} + \left[(A^{\circ}_{1x} \nu_x + A^{\circ}_{1y} \nu_y + A^{\circ}_{1z} \nu_z) \mathcal{P}_z \right]^{(2)} \right\}$$

$$dR_{2x} : \quad \{ [(A^{\circ}_{1x} \tau_x + A^{\circ}_{1y} \tau_y + A^{\circ}_{1z} \tau_z) (- L_x)]^{(1)}$$

$$+ [(A^{\circ}_{1x} v_x + A^{\circ}_{1y} v_y + A^{\circ}_{1z} v_z) (- L_x)]^{(2)} \}$$

$$dR_{2y} : \quad \{ [(A^{\circ}_{1x} \tau_x + A^{\circ}_{1y} \tau_y + A^{\circ}_{1z} \tau_z) (- L_y)]^{(1)}$$

$$+ [(A^{\circ}_{1x} v_x + A^{\circ}_{1y} v_y + A^{\circ}_{1z} v_z) (- L_y)]^{(2)} \}$$

$$dR_{2z} : \quad \{ [(A^{\circ}_{1x} \tau_x + A^{\circ}_{1y} \tau_y + A^{\circ}_{1z} \tau_z) (- L_z)]^{(1)}$$

$$+ [(A^{\circ}_{1x} v_x + A^{\circ}_{1y} v_y - A^{\circ}_{1z} v_z) (- L_z)]^{(2)} \}$$

$$dI_{1x}^{(1)} : \quad a_1^{(1)} (\tau_x t_{11} + \tau_y t_{21} + \tau_z t_{31})^{(1)} + (\tau_x A^{\circ}_{1x} + \tau_y A^{\circ}_{1y} + \tau_z A^{\circ}_{1z})^{(1)}$$

$$(M_{1x} t_{11} + M_{1y} t_{21} + M_{1z} t_{31})^{(1)}$$

$$dI_{1y}^{(1)} : \quad a_1^{(1)} (\tau_x t_{12} + \tau_y t_{22} + \tau_z t_{32})^{(1)} + (\tau_x A^{\circ}_{1x} + \tau_y A^{\circ}_{1y} + \tau_z A^{\circ}_{1z})^{(1)}$$

$$(M_{1x} t_{12} + M_{1y} t_{22} + M_{1z} t_{32})^{(1)}$$

$$dI_{2x}^{(1)} : \quad (\tau_x A^{\circ}_{1x} + \tau_y A^{\circ}_{1y} + \tau_z A^{\circ}_{1z})^{(1)} (N_{1x} t_{11} + N_{1y} t_{21} + N_{1z} t_{31})^{(1)}$$

$$dI_{2y}^{(1)} : \quad (\tau_x A^{\circ}_{1x} + \tau_y A^{\circ}_{1y} + \tau_z A^{\circ}_{1z})^{(1)} (N_{1x} t_{12} + N_{1y} t_{22} + N_{1z} t_{32})^{(1)}$$

$$dI_{1x}^{(2)} : \quad a_1^{(2)} (v_x t_{11} + v_y t_{21} + v_z t_{31})^{(2)} + (v_x A^{\circ}_{1x} + v_y A^{\circ}_{1y} + v_z A^{\circ}_{1z})^{(2)}$$

$$(M_{1x} t_{11} + M_{1y} t_{21} + M_{1z} t_{31})^{(2)}$$

$$dI_{1y}^{(2)} : \quad a_1^{(2)} (v_x t_{12} + v_y t_{22} + v_z t_{32})^{(2)} + (v_x A^{\circ}_{1x} + v_y A^{\circ}_{1y} + v_z A^{\circ}_{1z})^{(2)}$$

$$(M_{1x} t_{12} + M_{1y} t_{22} + M_{1z} t_{32})^{(2)}$$

$$dI_{2x}^{(2)} : \quad (v_x A^{\circ}_{1x} + v_y A^{\circ}_{1y} + v_z A^{\circ}_{1z})^{(2)} (N_{1x} t_{11} + N_{1y} t_{21} + N_{1z} t_{31})^{(2)}$$

$$dI_{2y}^{(2)} : \quad (v_x A^{\circ}_{1x} + v_y A^{\circ}_{1y} + v_z A^{\circ}_{1z})^{(2)} (N_{1x} t_{12} + N_{1y} t_{22} + N_{1z} t_{32})^{(2)}$$

$$\text{CONSTANT TERM} = -\left[R_{1X}K_X + (A_{1X}^{(1)})^2 - (A_{1X}^{(2)})^2 + R_{1Y}K_Y + (A_{1Y}^{(1)})^2 - (A_{1Y}^{(2)})^2 + R_{1Z}K_Z + (A_{1Z}^{(1)})^2 - (A_{1Z}^{(2)})^2\right]$$

10. EQUATIONS 101, 102, 103 - BLOCK ADJUSTMENT

Four exposure stations and the corresponding images of a ground point on each photograph are involved. Considered simultaneously they express the condition of intersection at one point of the four rays $\bar{A}_1, \bar{A}_2, \bar{A}_1^*, \bar{A}_2^*$ from the four perspective centers to the corresponding four images.

a. Equation 101 Coefficients

dT_{1X} : $A_{1X}^* G_X$

dT_{1Y} : $A_{1X}^* G_Y + a_1 A_{1Z}^*$

dT_{1Z} : $A_{1X}^* G_Z - a_1 A_{1Y}^*$

dR_{1X} : $A_{1X}^* L_X + 1$

dR_{1Y} : $A_{1X}^* L_Y$

dR_{1Z} : $A_{1X}^* L_Z$

dT_{2X} : $A_{1X}^* P_X$

dT_{2Y} : $A_{1X}^* P_Y$

dT_{2Z} : $A_{1X}^* P_Z$

dR_{2X} : $- A_{1X}^* L_X$

dR_{2Y} : $- A_{1X}^* L_Y$

dR_{2Z} : $- A_{1X}^* L_Z$

$dT_{1X}^* : \quad -A_{1X}^{o*} G_X^*$

$dT_{1Y}^* : \quad -(A_{1X}^{o*} G_Y^* + a_1^* A_{1Z}^{o*})$

$dT_{1Z}^* : \quad -(A_{1X}^{o*} G_Z^* - a_1^* A_{1Y}^{o*})$

$dR_{1X}^* : \quad -(A_{1X}^{o*} L_X^* + 1)$

$dR_{1Y}^* : \quad -A_{1X}^{o*} L_Y^*$

$dR_{1Z}^* : \quad -A_{1X}^{o*} L_Z^*$

$dT_{2X}^* : \quad -A_{1X}^{o*} P_X^*$

$dT_{2Y}^* : \quad -A_{1X}^{o*} P_Y^*$

$dT_{2Z}^* : \quad -A_{1X}^{o*} P_Z^*$

$dR_{2X}^* : \quad A_{1X}^{o*} L_X^*$

$dR_{2Y}^* : \quad A_{1X}^{o*} L_Y^*$

$dR_{2Z}^* : \quad A_{1X}^{o*} L_Z^*$

$dI_{1X} : \quad a_1 t_{11} + A_{1X}^o (M_{1X} t_{11} + M_{1Y} t_{21} + M_{1Z} t_{31})$

$dI_{1Y} : \quad a_1 t_{12} + A_{1X}^o (M_{1X} t_{12} + M_{1Y} t_{22} + M_{1Z} t_{32})$

$dI_{2X} : \quad A_{1X}^o (N_{1X} t'_{11} + N_{1Y} t'_{21} + N_{1Z} t'_{31})$

$dI_{2Y} : \quad A_{1X}^o (N_{1X} t'_{12} + N_{1Y} t'_{22} + N_{1Z} t'_{32})$

$dI_{1X}^* : \quad -a_1^* t_{11}^* - A_{1X}^{o*} (M_{1X}^* t_{11} + M_{1Y}^* t_{21}^* + M_{1Z}^* t_{31}^*)$

dI_{1Y}^{*} : $- a_1^{*} t_{12}^{*} - A_{1X}^{\circ *} (M_{1X}^{*} t_{12}^{*} + M_{1Y}^{*} t_{22}^{*} + M_{1Z}^{*} t_{32}^{*})$

dI_{2X}^{*} : $- A_{1X}^{\circ *} (N_{1X}^{*} t'^{*}_{11} + N_{1Y}^{*} t'^{*}_{21} + N_{1Z}^{*} t'^{*}_{31})$

dI_{2Y}^{*} : $- A_{1X}^{\circ *} (N_{1X}^{*} t'^{*}_{12} + N_{1Y}^{*} t'^{*}_{22} + N_{1Z}^{*} t'^{*}_{32})$

CONSTANT TERM $= R_{1X}^{*} + A_{1X}^{*} - R_{1X} - A_{1X}$

b. Equation 102 Coefficients

dT_{1X} : $A_{1Y}^{\circ} G_X - a_1 A_{1Z}^{\circ}$

dT_{1Y} : $A_{1Y}^{\circ} G_Y$

dT_{1Z} : $A_{1Y}^{\circ} G_Z + a_1 A_{1X}^{\circ}$

dR_{1X} : $A_{1Y}^{\circ} L_X$

dR_{1Y} : $- (A_{1Y}^{\circ} L_Y + 1)$

dR_{1Z} : $A_{1Y}^{\circ} L_Z$

dT_{2X} : $A_{1Y}^{\circ} P_X$

dT_{2Y} : $A_{1Y}^{\circ} P_Y$

dT_{2Z} : $A_{1Y}^{\circ} P_Z$

dR_{2X} : $- A_{1Y}^{\circ} L_X$

dR_{2Y} : $-A_{1Y}^{o} L_Y$

dR_{2Z} : $-A_{1Y}^{o} L_Z$

dT_{1X}^{*} : $-(A_{1Y}^{o*} G_X^{*} - d_1^{*} A_{1Z}^{o*})$

dT_{1Y}^{*} : $-A_{1Y}^{o*} G_Y^{*}$

dT_{1Z}^{*} : $-(A_{1Y}^{o*} G_Z^{*} + d_1^{*} A_{1X}^{o*})$

dR_{1X}^{*} : $-A_{1Y}^{o*} L_X^{*}$

dR_{1Y}^{*} : $-(A_{1Y}^{o*} L_Y^{*} + 1)$

dR_{1Z}^{*} : $-A_{1Z}^{o*} L_Z^{*}$

dT_{2X}^{*} : $-A_{1Y}^{o*} P_X^{*}$

dT_{2Y}^{*} : $-A_{1Y}^{o*} P_Y^{*}$

dT_{2Z}^{*} : $-A_{1Y}^{o*} P_Z^{*}$

dR_{2X}^{*} : $A_{1Y}^{*} L_X^{*}$

dR_{2Y}^{*} : $A_{1Y}^{o*} L_Y^{*}$

dR_{2Z}^{*} : $A_{1Y}^{o*} L_Z^{*}$

dI_{1X} : $\theta_1 t_{21} + A_{1Y}^{o} (M_{1X} t_{11} + M_{1Y} t_{21} + M_{1Z} t_{31})$

dI_{1Y} : $\theta_1 t_{22} + A_{1Y}^{o} (M_{1X} t_{12} + M_{1Y} t_{22} + M_{1Z} t_{32})$

dI_{2X} : $A_{1Y}^{o} (N_{1X} t'_{11} + N_{1Y} t'_{21} + N_{1Z} t'_{31})$

dI_{2Y} : $A^{\circ}_{1Y}(N_{1X}t'_{12} + N_{1Y}t'_{22} + N_{1Z}t'_{32})$

dI^{*}_{1X} : $-a_1^{*}t^{*}_{21} - A^{\circ *}_{1Y}(M^{*}_{1X}t^{*}_{11} + M^{*}_{1Y}t^{*}_{21} + M^{*}_{1Z}t^{*}_{31})$

dI^{*}_{1Y} : $-a_1^{*}t^{*}_{22} - A^{\circ *}_{1Y}(M^{*}_{1X}t^{*}_{12} + M^{*}_{1Y}t^{*}_{22} + M^{*}_{1Z}t^{*}_{32})$

dI^{*}_{2X} : $-A^{\circ *}_{1Y}(N^{*}_{1X}t'^{*}_{11} + N^{*}_{1Y}t'^{*}_{21} + N^{*}_{1Z}t'^{*}_{31})$

dI^{*}_{2Y} : $-A^{\circ *}_{1Y}(N^{*}_{1X}t'^{*}_{12} + N^{*}_{1Y}t'^{*}_{22} + N^{*}_{1Z}t'^{*}_{32})$

CONSTANT TERM = $R^{*}_{1Y} + A^{*}_{1Y} - R_{1Y} - A_{1Y}$

c. Equation 103 Coefficients

dT_{1X} : $A^{\circ}_{1Z}G_X + a_1 A^{\circ}_{1Y}$

dT_{1Y} : $A^{\circ}_{1Z}G_Y - a_1 A^{\circ}_{1X}$

dT_{1Z} : $A^{\circ}_{1Z}G_Z$

dR_{1X} : $A^{\circ}_{1Z}L_X$

dR_{1Y} : $A^{\circ}_{1Z}L_Y$

dR_{1Z} : $A^{\circ}_{1Z}L_Z + 1$

dT_{2X} : $A^{\circ}_{1Z}P_X$

dT_{2Y} : $A^{\circ}_{1Z}P_Y$

dT_{2Z} : $A_{1Z}^{\circ} P_Z$

dR_{2X} : $-A_{1Z}^{\circ} L_X$

dR_{2Y} : $-A_{1Z}^{\circ} L_Y$

dR_{2Z} : $-A_{1Z}^{\circ} L_Z$

dT_{1X}^{*} : $-(A_{1Z}^{\circ *} G_X^{*} + a_1^{*} A_{1Y}^{\circ *})$

dT_{1Y}^{*} : $-(A_{1Z}^{\circ *} G_Y^{*} - a_1^{*} A_{1X}^{\circ *})$

dT_{1Z}^{*} : $-A_{1Z}^{\circ *} G_Z^{*}$

dR_{1X}^{*} : $-A_{1Z}^{\circ *} L_X^{*}$

dR_{1Y}^{*} : $-A_{1Z}^{\circ *} L_Y^{*}$

dR_{1Z}^{*} : $-(A_{1Z}^{\circ *} L_Z^{*} + 1)$

dT_{2X}^{*} : $-A_{1Z}^{\circ *} P_X^{*}$

dT_{2Y}^{*} : $-A_{1Z}^{\circ *} P_Y^{*}$

dT_{2Z}^{*} : $-A_{1Z}^{\circ *} P_Z^{*}$

dR_{2X}^{*} : $A_{1Z}^{\circ *} L_X^{*}$

dR_{2Y}^{*} : $A_{1Z}^{\circ *} L_Y^{*}$

dR_{2Z}^{*} : $A_{1Z}^{\circ *} L_Z^{*}$

dI_{1X} : $a_1 t_{31} + A_{1Z}^{\circ} (M_{1X} t_{11} + M_{1Y} t_{21} + M_{1Z} t_{31})$

dI_{1Y} : $a_1 t_{32} + A_{1Z}^{-}(M_{1X} t_{12} + M_{1Y} t_{22} + M_{1Z} t_{32})$

dI_{2X} : $A_{1Z}^{-}(N_{1X} t'_{11} + N_{1Y} t'_{21} + N_{1Z} t'_{31})$

dI_{2Y} : $A_{1Z}^{-}(N_{1X} t'_{12} + N_{1Y} t'_{22} + N_{1Z} t'_{32})$

dI_{1X}^{*} : $-a_1^{*} t_{31}^{*} - A_{1Z}^{-*}(M_{1X}^{*} t_{11}^{*} + M_{1Y}^{*} t_{21}^{*} + M_{1Z}^{*} t_{31}^{*})$

dI_{1Y}^{*} : $-a_1^{*} t_{32}^{*} - A_{1Z}^{-*}(M_{1X}^{*} t_{12}^{*} + M_{1Y}^{*} t_{22}^{*} + M_{1Z}^{*} t_{32}^{*})$

dI_{2X}^{*} : $-A_{1Z}^{-*}(N_{1X}^{*} t'^{*}_{11} + N_{1Y}^{*} t'^{*}_{21} + N_{1Z}^{*} t'^{*}_{31})$

dI_{2Y}^{*} : $-A_{1Z}^{-*}(N_{1X}^{*} t'^{*}_{12} + N_{1Y}^{*} t'^{*}_{22} + N_{1Z}^{*} t'^{*}_{32})$

CONSTANT TERM $= R_{1Z}^{*} + A_{1Z}^{*} - R_{1Z} - A_{1Z}$

11. EQUATIONS 8, 9, 10 - EXPOSURE STATION OF KNOWN ϕ, λ and h

They merely express dR_x, dR_y, and dR_z in terms of the unknown residuals $d\phi$, $d\lambda$, and dh.

a. Equation 8 Coefficients

dT_x : 0

dT_y : 0

dT_z : 0

dR_x : 1

dR_y : 0

dR_z : 0

$d\phi$: $\left[a(1-e^2)(1-e^2\sin^2\phi)^{-3/2} + h\right]\sin\phi\cos\lambda$

$d\lambda$: $\left[a(1-e^2\sin^2\phi)^{-1/2} + h\right]\cos\phi\sin\lambda$

dh : $-\cos\phi\cos\lambda$

CONSTANT TERM = 0

b. Equation 9 Coefficients

dT_x : 0

dT_y : 0

dT_Z : 0

dR_X : 0

dR_Y : 1

dR_Z : 0

$d\phi$: $[a(1-e^2)(1-e^2\sin^2\phi)^{-3/2} + h] \sin\phi \sin\lambda$

$d\lambda$: $-[a(1-e^2\sin^2\phi)^{-1/2} + h] \cos\phi \cos\lambda$

dh : $-\cos\phi \sin\lambda$

CONSTANT TERM = 0

c. Equation 10 Coefficients

dT_X : 0

dT_Y : 0

dT_Z : 0

dR_X : 0

dR_Y : 0

dR_Z : 1

$d\phi$: $-[a(1-e^2)(1-e^2\sin^2\phi)^{-3/2} + h] \cos\phi$

$d\lambda$: 0

dh : $-\sin\phi$

CONSTANT TERM = 0

12. EQUATION 11 - EXPOSURE STATION OF KNOWN λ

The perspective center is forced to lie in the plane of equal longitude λ.

Coefficients

dT_x : 0

dT_y : 0

dT_z : 0

dR_x : $\sin\lambda$

dR_y : $-\cos\lambda$

dR_z : 0

$d\lambda$: $\cos\lambda\, R_x + \sin\lambda\, R_y$

CONSTANT TERM $= \cos\lambda\, R_y - \sin\lambda\, R_x$

13. EQUATION 12 - EXPOSURE STATION OF KNOWN LATITUDE ϕ AND LONGITUDE λ

The perspective center is required to lie in the prime vertical plane.

Auxiliary Definitions

$$\gamma = \cos\lambda \sin\phi$$

$$\delta = \sin\lambda \sin\phi$$

$$\epsilon = -\cos\phi$$

$$\zeta = \frac{de^2 \sin\phi \cos\phi}{\sqrt{1-e^2 \sin^2\phi}}$$

Coefficients

dT_X : 0

dT_Y : 0

dT_Z : 0

dR_X : γ

dR_Y : δ

dR_Z : ϵ

$d\lambda$: $-\delta R_X + \gamma R_Y$

$d\phi$: $\cos\lambda \cos\phi R_X + \sin\lambda \cos\phi R_Y + \sin\phi R_Z$

$$-de^2 \frac{\cos^2\phi - \sin^2\phi \; (1-e^2 \sin^2\phi)}{(1-e^2 \sin^2\phi)^{3/2}}$$

CONSTANT TERM $= \zeta - [\gamma R_X + \delta R_Y + \epsilon R_Z]$

14. EQUATION 12A - EXPOSURE STATION OF KNOWN LATITUDE ϕ

The perspective center is forced to lie on the cone of equal latitude ϕ.

Auxiliary Definitions

$$t = \frac{ae^2 \sin\phi}{\sqrt{1-e^2\sin^2\phi}}$$

Coefficients

dT_X : 0

dT_Y : 0

dT_Z : 0

dR_X : $2R_X$

dR_Y : $2R_Y$

$$dR_Z : \frac{-2(R_Z + t)}{\tan^2\phi}$$

$$d\phi : \frac{2(R_Z + t)}{\tan^2\phi}\left[\frac{R_Z + t}{\sin\phi\cos\phi} - \frac{ae^2\cos\phi}{(1-e^2\sin^2\phi)^{3/2}}\right]$$

$$\text{CONSTANT TERM} = \frac{(R_Z + t)^2}{\tan^2\phi} - \left[R_X^2 + R_Y^2\right]$$

15. EQUATION 13 - EXPOSURE STATION OF KNOWN ELEVATION

It enforces the perspective center to lie on a sphere defined similarly to Equation 7.

Auxiliary Definitions

$$n = \frac{a}{\sqrt{1-e^2 \sin^2 \phi}}$$

$$t = \frac{a e^2 \sin \phi}{\sqrt{1-e^2 \sin^2 \phi}}$$

Coefficients

dT_X : 0

dT_Y : 0

dT_Z : 0

dR_X : $\dfrac{R_X}{(n+h)}$

dR_Y : $\dfrac{R_Y}{(n+h)}$

dR_Z : $\dfrac{R_Z + t}{(n+h)}$

dh : -1

$$\text{CONSTANT TERM} = \frac{1}{2}\left\{ n + h - \frac{[R_X^2 + R_Y^2 + (R_Z + t)^2]}{(n+h)} \right\}$$

VIII. OPERATING INSTRUCTIONS

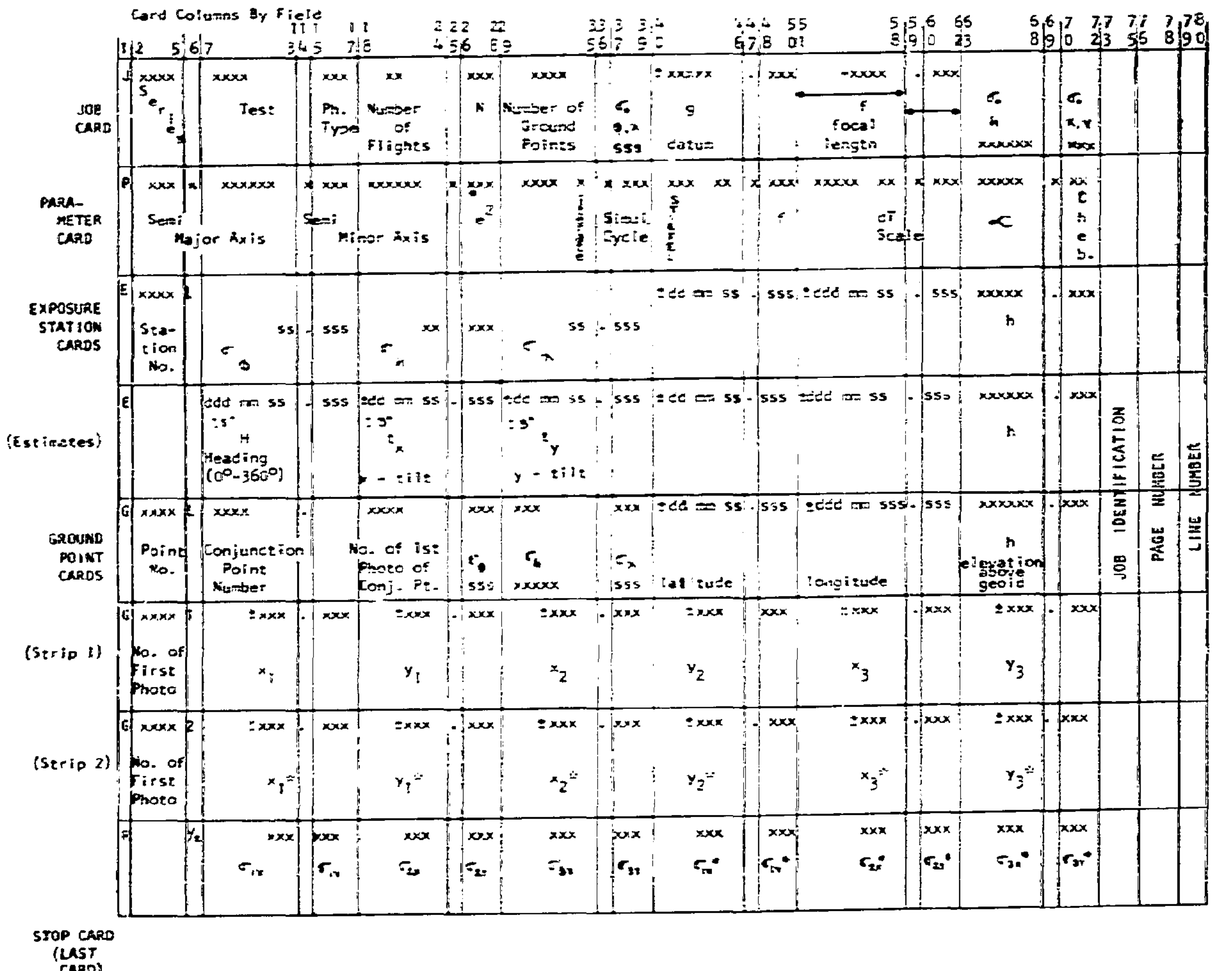
A. Loading Sequence

The program, control cards, and input data are written on magnetic tape in the following sequence:

1. ID Control Card (monitor accounting)
2. XEQ Control Card
3. Chain 1 Control Card
4. Segment 1 Binary Deck
5. Chain 2 Control Card
6. Segment 2 Binary Deck
7. Chain 23 Control Card
8. Segment 23 Binary Deck
9. Chain 3 Control Card
10. Segment 3 Binary Deck
11. Chain 4 Control Card
12. Segment 4 Binary Deck
13. Data Control Card
14. Data Deck - Binary Coded Decimal
    a. J - Job Card
    b. P - Parameter Card
    c. E - Exposure Stations
    d. G - Ground Control Points
    e. S - Stop Card

FIGURE 1

KEY: dd mm ss represents digits of degrees, minutes, and seconds
xxxx represents digits of a number, including leading zeroes
L represents a letter code for the type of point (Col. 6)



CARD INPUT FORMATS

B. Magnetic Tape Usage

In addition to the tape units used by the Fortran Monitor System, the following tapes are used for temporary storage by the object program.

A4 - Segments 1 and 2

B1 - Condition Equation Sets

B2 - Segments 23, 3, and 4

B3 - Exposure Station unit variance, ground point data, VECDEX, MATDEX

All output is put on tape A3 for off-line printing and punching.

When the run is completed either by normal exit or after error exit, control is returned to FMS.

C. Input Format Guide

1. Summary

Figure 1 displays the eight types of "IBM" cards on which the input data for a block or strip triangulation must be punched. The card are then arranged in a deck, as follows:

Job Card

Parameter Card

All Exposure Station Card sets

All Ground Point Card sets

Stop Card

The following paragraphs discuss the contents of each card type in detail, referring to successive lines of Figure 1.

2. Job Card (Line 1)

The computer identifies the job card by the letter J

in column 1. The numbers of the series and test, type of photography, and number of flights are read and stored by the computer. However, they have no significance for the actual triangulation process, and may be left blank.

N is the number of exposure stations or photographs. After all cards have been read, N and the number of ground points (appearing in columns 29-35) will be compared with the actual tallies, and an error stop will occur if too few or too many exposure stations or ground point cards were read.

The unit standard deviation of the ϕ and λ for exposure stations and ground points is defined in column 37-39 in thousandths of seconds of arc.

The datum g gives the altitude in feet, relative to the given spheroid, of the surface from which all altitudes are given. If columns 40-50 are left blank, the program will assume that all altitudes in the input data are corrected to the reference spheroid.

The focal length f will be taken as the "z" coordinate, in the photographic model, corresponding to the measured values of x and y which give the position of a ground point on a photograph. Thus, x, y, and f must have the same dimension, e.g. all inches or all centimeters. The negative sign of f is necessary for correct orientation.

The unit standard deviation of the altitude (h) used for the exposure stations and ground points is specified in columns 63-68 in thousandths of feet.

The unit standard deviation to be used for the photographic x and y coordinates is defined in columns 70-72 in microns.

Columns 73-80 of all cards are not read by the computer, and the divisions shown are only a suggestion. Another possibility is to use them to give consecutive numbers to every card of a deck, so that its original order can be restored in case of accident.

J Card Format

σ_o	in	sec $\times 10^{-3}$	Col.	37	39	(ϕ, λ)
σ_o		ft $\times 10^{-3}$		63	68	(h)
σ_o		microns		70	72	(x,y)

E Card Format

σ	in	sec	Col.	7	12	(ϕ)
σ		sec $\times 10^{-3}$		15	17	(ϕ)
σ		ft		18	24	(h)
σ		ft $\times 10^{-3}$		26	28	(h)
σ		sec		29	35	(λ)
σ		sec $\times 10^{-3}$		37	39	(λ)

G Card Format

First Card

σ	in	sec $\times 10^{-3}$	Col.	26	28	(ϕ)
σ		ft. $\times 10^{-3}$		29	35	(h)
σ		sec $\times 10^{-3}$		37	39	(λ)

R Card Format

anything in	microns	Col.	2	6
σ_{1x}	microns		7	13
σ_{1y}	microns		15	17
σ_{2x}	microns		18	24
σ_{2y}	microns		26	28
σ_{3x}	microns		29	35
σ_{3y}	microns		37	39
σ_{1x}^*	microns		40	46
σ_{1y}^*	microns		48	50
σ_{2x}^*	microns		51	58
σ_{2y}^*	microns		60	62
σ_{3x}^*	microns		63	68
σ_{3y}^*	microns		70	72

PHYSICAL UNITS OF INPUT DATA

3. Parameter Card (line 2)

	Format
Column 1:	P
Columns 3-12:	integral part of 1024a, where a = semi-major axis of reference ellipsoid in feet (standard value of a = 20,925,832.16)
Columns 14-23:	integral part of 1024b, semi-major axis (standard value of b = 20,854,892.02)
Columns 25-32:	integral part of 2^{35} e^2, where $e^2 = 1 - b^2/a^2$ (2^{35} = 34,359,738,368) (standard value of e^2 = .006,768,658,0)
Column 34:	number of iterations before resection and break-point stop (pressing the START button will initiate another set of iterations) (standard value 4) using 3
Columns 36-39:	number of allowable cycles of Simultaneous Solution, assuming continued but slow convergence.
Columns 41-43:	cycles of Stiefel method, in each batch, between convergence tests (standard value 6 x no. of photos)
Columns 45-55:	convergence criterion f, where Simultaneous Solution is assumed correct if for every normal equation constant term k_i, the corresponding residual r_i satisfies $\|r_i\| < f \|k_i\|$; f must be greater than .000,000,05 in any case (decimal point and up to 10 digits) (standard value .000,005)
Columns 57-62:	scale factor for dT coefficients; coefficients are divided by this factor before normalization of the condition equation, so that the dT values from the Simultaneous Solution must be divided by it also (division occurs before printing) (factor is an integer) (standard value 1)
Columns 64-69:	ratio c for computing artificial lower bound for Chebyshev portion of Simultaneous Solution (upper bound b is computed from normal matrix, Chebyshev

method is then set for eigenvalues in the range cb to b) (decimal point and up to 5 digits) (standard value .02)

Columns 71-72: number of cycles n_c of Chebyshev method; for a normal accuracy a,

$$n_c = \text{greatest integer in } \frac{\cosh^{-1} a}{\cosh^{-1} \frac{1+c}{1-c}}$$

(standard value 50)

General

Any digits punched in the columns listed will be taken as new values, to be substituted for the standard cases. Notice that all values except the convergence criterion f and the ratio c are pure integers with no sign or decimal point, while f and c each contain a decimal point and may (but usually will not) have digits or zeros to the left of the decimal point. No spaces should be left between digits.

4. Exposure Station Cards (lines 3 and 4)

A pair of cards, each with the letter E in column 1, must be made up for each exposure station. Columns 2 through 5 of the first card (and the second, if desired) contain the number of the photographs. Photographs must be numbered consecutively, and in the same direction on each strip of a block. The number of the first photograph may be 1 or larger, but not zero.

Column 6 contains the exposure station type letter as taken from Figure 3.

Columns 7-13 contain the integer portion of the standard deviation used for the exposure station latitude ϕ in seconds of arc.

Columns 15-17 contain the fractional portion of the standard deviation used for the exposure station latitude ϕ in thousandths of seconds of arc.

Columns 18-24 define the integer portion of the standard deviation used for the exposure station altitude (h) in feet.

Columns 26-28 define the fractional portion of the standard deviation used for the exposure station altitude (h) in thousandths of feet.

Columns 29-35 contain the integer portion of the standard deviation for the exposure station longitude λ given in seconds of arc.

Columns 37-39 define the fractional portion of the standard deviation for the exposure station longitude λ given in thousandths of seconds of arc.

Columns 40-50, 51-62, and 63-72 contain such data as is consistent with the type; if a value is not known, it is left blank. Note that latitude, ϕ, ranges from $-90^{\circ}00'00.000''$ to $+90^{\circ}00'00.000''$, while longitude ranges from $-180^{\circ}00'00.000''$ (West) to $+180^{\circ}00'00.000''$ (East). Zeros for minutes and seconds must be punched on the card; these are "trailing zeros" which establish the degrees-minutes-seconds relationship. Leading zeros, however, need not be punched; for instance, $-00^{\circ}03'10.567''$ could have simply -310 anywhere in the field preceding the decimal point. Minus signs, if used, must not be separated from the leading digit, nor may digits be separated by blank columns within a field.

Note that decimal points may appear only in certain columns (14, 25, 36, 47, 59, 69). If desired, decimal points may be gang-punched in all cards in these columns, even where (as in N on the Job Card) the "decimals" have no relation to the "integer" to the left.

The reader should also remember to always write three figures in the decimal positions if they are in fact part of a number to the left, since (for all angles and elevations) conversion involves division of the number in these columns, taken as an integer, by 1000.

The second card for each exposure station contains the estimated heading, tilt, and geographic location of the photograph's perspective center. None of this information may be omitted. If ϕ, λ, and/or h differ from values on the preceding card, the "estimates" will be used in the preparation of the orientation matrix (M).

5. Ground Point Cards (Lines 5,6,7, and 8)

Each ground point is represented in the input deck by two or three cards containing a G in column 1. The number of the point appears on the first card; it will be used to identify the point in the output. The order in which ground points are processed, for instance in forming condition equations, is the same as their order in the input deck.

Column 6 of the first card shows the type of point as taken from Figure 3.

The conjunction point number, and number of first photograph of conjunction point, are taken (if present) as a requent for Condition Equation 70. This equation is not otherwise computed. It involves two ground points appearing on the same pair of photographs. The specified conjunction point must follow (but not necessarily be next) the current point in the input deck. The pair of photographs must, of course, be consecutive on a particular strip; they could, however, be in different locations with respect to the ground points, e.g., as photos 2 and 3 of strip 2 for the first point, and photos 1 and 2 of strip 1 for the second

point. The cards for the specified conjunction point do not have any special marks of identification; if numbers appear in columns 7-13 and 15-24 of the first card for that point, they must refer to a second application of Equation 70 with a third point further on in the deck.

Columns 26-28 contain the standard deviation for the ground point latitude given in thousandths of seconds of arc (ϕ).

Columns 29-35 define the standard deviation for the ground point altitude (h) given in thousandths of feet.

Columns 37-39 contain the standard deviation for the ground point longitude (λ) given in thousandths of seconds of arc.

INFORMATION CODE FOR GROUND POINTS AND EXPOSURE STATIONS

Known Information	Letter Code
latitude, longitude, elevation	A
latitude, longitude, elevation (not of equal quality)	B
latitude and longitude	C
latitude and elevation	D
latitude	E
longitude and elevation	F
longitude	G
elevation	H
none	K
computed in resection but not used for photo orientation	L

Figure 3. Type Codes

Geographic coordinates appear if and only if the type letter of column 6 implied that some or all of them were known.

The second card for a ground point contains an identifying "1" (for strip 1) in column 6. X_1 and y_1 are the measurements (normally in inches) taken from the photograph whose number is given in columns 2-5. X_2, y_2 and x_3, y_3 are left blank, or contain measurements of the same point from the photographs which follow in the strip, i.e., if the first photo is No. 7, x_2 and y_2 must be taken from photo No. 8, and x_3, y_3 from No. 9. If a point happens to lie at (0,0) on the photograph, at least one zero must be punched in each position for x and y. If x_2 and y_2 are blank, the computer will not look for x_3 and y_3.

The sign, digits, and decimal point in columns 7-15, 18-24, etc. may be separated by blanks in the case of x-y measurements. Plus signs need not appear (this also applies to all signed data, on exposure station cards as well).

A third card, identified by the G in column 1 and a 2 in column 6, appears only if data for a second strip is available. Photo 1 of strip 2 must then overlap photo 1 of strip 1, and so forth. Unless the ground point is of type A or C, data from a single photo in the second strip cannot be used and should be left out ot avoid stalling the computer in the middle of the Condition Equation Former.

Strip 1 may lie either above or below strip 2 for any ground point, the only difference being that condition equations 5, 6, 6A, and/or 7 are computed <u>only</u> for the first two photos of the first strip.

Each set of ground point cards must be terminated with an R card. This card specified the standard deviations to be used for the

x and y coordinates as defined by the following format:

Column 1 R

11-13	photograph 1	strip 1
15-17	photograph 1	strip 1
22-24	photograph 2	strip 1
26-28	photograph 2	strip 1
33-35	photograph 3	strip 1
37-39	photograph 3	strip 1
44-46	photograph 1	strip 2
48-50	photograph 1	strip 2
56-58	photograph 2	strip 2
60-62	photograph 2	strip 2
66-68	photograph 3	strip 2
70-72	photograph 3	strip 2

6. Stop Card (not shown)

This card contains an S in column 1. Any other data on the card is ignored.

D. <u>Output Format Guide</u>

1. Cycle Output

On every cycle including the last, the corrected values of (M) and $\overline{R}$ are printed in the following format:

title line:		Photo	(No.)
data lines:	(1)	R_x	M_{11} M_{12} M_{12}
	(2)	R_y	M_{21} M_{22} M_{23}
	(3)	R_z	M_{31} M_{32} M_{33}

The first series of this output shows the initial values prepared from estimates and contains, in addition, the type letter, radian values of phi and lamda, and altitude on the title line. Matrix values are printed in floating point, as a signed power of 10 followed by a signed mantissa of eight digits. For instance, the numbers 0.0987, 0.987 and -98,765,432,000.0 are printed as:

-1 98700000

0 98700000

and 11-98765432.

2. Final Output

After cycle output on the last cycle, the heading, swing, azimuth, and tilt are computed for each photograph, and geographic coordinates are computed for all exposure stations and ground points. Angular data are printed as degrees, minutes, and seconds, the three numbers being separated by spaces: for instance, 146°8'40.355" is printed 146 8 40.355. The first list is for exposure stations:

title line: PHOTO(No.)

data line: H s a t t_x t_y ϕ λ h

This is followed by the list of ground points:

title line: GROUND POINT(No.) TYPE(t) KNOWN (ϕ) (λ) (h)

data lines: CALCULATED ϕ λ h Δx Δy Δz

In the title line, following "KNOWN", is the control data as given in the input on the first card for this ground point; if any values were not given, they are replaced by single x's. The calculated values of ϕ, λ, and h are invariably obtained as follows. First, values of x, y, and z computed for each pair of photographs on which the point appears. These are

averaged, and the average values used to compute ϕ, λ, and h. The differences (in absolute value) between the average values of x, y, and z, and the values which are furthest out of line (the maximum deviates) are printed as Δx, Δy, and Δz.

3. Matrix Solution

On each cycle including the last, the solution vector to the normal matrix is printed in floating point, its elements in the order dT_{1x}, dT_{1y}, dT_{1z}, dR_{1x}, dR_{1y}, dR_{1z}, dT_{2x}, etc. through dR_{nz}.

4. Condition Equation Coefficients

During Segment 2 the coefficients of each condition equation for each exposure station or ground point will be printed as soon as they are formed and have been treated by the weight former:

title line: COEFFS. OF COND. EQ. CALLED FROM (nnnnn)
PHOTOS (i) (j) (k) (l)

data lines: (Coefficients of dT_{ix}, dT_{i}, ... , dR_{iz}, dT_{jx}, ... , dR_{jz}, ... , dR_{lz}, d_{ix}, d_{iz}, ... d_{ex}, d_{ey}, $d\phi$, $d\lambda$, dh and constant term, all in floating point)

The octal number (nnnnn) in the title line refers to a line in the Condition Equation Former program. A list of these numbers can be made up by inspecting the list produced by FMS on compilation of Segment 2, and noting all references to equation former subroutines.

5. Normal Matrix and Constant Vector

A suitable title line will be printed, followed by the 36(N+K) elements of the normal matrix, the 6N elements of the constant vector, and the sum of the squares of the constant vector (k^2). The

matrix elements are printed in 36-word blocks, beginning with the N main-diagonal blocks (in the same order as the terms of the vectors - see 4. Matrix Solution above) followed by the K non-vanishing blocks in the upper triangle of the matrix, in order row by row.

6. "Data Processing" Output

Segment 1, the following data will be printed for each ground point:

GROUND PT, (No.) (Type) (Ø radians) (radians) (h)
(X) (Y) (Octal Equation 70 code word)

(Octal Strip 1 index word) (a_{1x}) (a_{1y}) (a_{1z}) (a_{2x}) ... (a_{3z})

(Octal Strip 2 index word) (a_{1x}^{*}) . . . . (a_{3z}^{*})

Of this data, , , and h are printed only if given; X, Y, and Z are computed and printed only if , , and h are given; and the strip 2 data line is omitted if there is no Strip 2 input.

At completion of Segment 1, a single line on the attached printer gives N, 36(N+K), and T, i.e., the number of photos, total matrix storage, and number of photo-pair linkages.

E. Error Routine

This program is designed in accordance with Standard FMS operating procedures. Accordingly, all error halts have been converted to a transfer to an error diagnostic routine: "ERROR". The error routine prints the contents of the following AC, MQ, SI, KEYS, XR1, XR2, XR4, when control reaches the TSX ERROR, 4 instruction. The contents of the AC and MQ are printed both in octal and decimal. The complement of the contents of XR1, XR2, XR4 are also printed. The decrement of the SI contains the complement of the address of the instruction which called the error routine.

Following this is a dump of the entire core which, joined with the

knowledge of the location of the error call, gives maximum offline debugging information.

The relocation constant used by the MIT-FMS system is 144 octal. Thus, if for example the line printed by the error routine gives as contents of the SI 07447700--------- the location of the TSX ERROR, 4 instruction is 3301-144=3135.

An error stop in Segment 2 called CEALR prints before the error routine ERROR is called the following information: SEGMENT 2 ALARM Q & P bits, AC, MQ, XR1, XR2, XR4.

The content of XR4 is the relocated complement of the location containing the last executed TSX instruction. Thus, if XR4=72437, the location of the TSX instruction is 5341-144=5175.

A stop in data reading could mean one of the following: bad job card, bad parameter card, no stop card, wrong number of E cards, wrong number of ground points. A stop in SEGMENT 2 ALARM (CEALR) is explained frequently by remarks in the coding.

IX. RESULTS AND CONCLUSIONS

The introduction of the least squares method into the analytical aerotriangulation problem for the adjustment of random errors gives, according to the theory of probability, the most probable solution. This solution consists of two parts: the first is formed by the location and orientation of the camera stations and the second is formed by the adjusted image points coordinates, the adjusted ground control coordinates, and, if any, the adjusted exposure stations control information.

The program in its present form computes only the first part of the solution. The reason for this is that to compute the residuals as given by the least squares solution requires additional computer time and storage area which can be justified only if high accuracy is desired. A discrepancy of say 10 microns in the image coordinates of a ground point is equivalent to a discrepancy of 0.3 m. or 1 ft. on the ground for a nominal scale of $\frac{1}{30,000}$. It is felt in this laboratory that errors of this order of magnitude can be tolerated as long as systematic errors corrections are not more precise than they are at the present time. The photogrammetric solution given by the program is then based on the exposure stations location and orientation as given by the least squares adjustment and on the unadjusted image point coordinates and control information.

To draw a general conclusion and evaluation of the program requires that a general testing plan, including fict tious data and specifically selected real data, be carried out. The tests conducted by this laboratory fall short of a conclusive testing plan, expecially as concerning the real

data. The reason for this is that no information other than the coordinates of the ground control points was available.

Before a short description and analysis of the tests conducted by this laboratory are presented, the units of the residuals will be discussed. The coefficients of the residuals in the condition equations are scaled so that the residuals of the image point coordinates are in microns. The residuals of angles and for ground and exposure station control are in units of 0.0206 seconds (scaling by 10^7), the residual of h is in units of feet, the weights are all dimensionless numbers, ratios of a reference variance to the respective variances of the quantities to be weighted.

A. FICTITIOUS DATA

1. Test 5

No. of photos:	6
Strips:	1
No. of ground points:	55

a.

Method of solution:	Stiefel
No. of iterations	4
No. of cycles per iteration:	2
Weights:	all one

Results

Gr. Pt.	Type	$\Delta\phi$ sec.	$\Delta\lambda$ sec.	Δh ft.
101	D	-0.003		-0.028
503	A	-0.003	-0.003	0.211
307	H			0.388
111	A	-0.002	0.001	-0.111
511	F		-0.001	0.117

where Δ = calculated - input

b.

Method of solution:	Stiefel
No. of iterations:	4
No. of cycles per iteration:	4
Weights:	all one

Results

Gr. PT.	Type	$\Delta\phi$	$\Delta\lambda$	Δh
101	D	0.000		0.147
503	A	0.000	-0.001	0.080
307	H			0.318
111	A	0.001	1.000	0.075
511	F	-0.001		0.059

To transform $\Delta\phi$ and $\Delta\lambda$ in feet, a rough estimate is $\frac{1}{100}$ sec. = 1 ft. for $\Delta\phi$ and $\frac{1}{100}$ sec. = 1 ft. x $\cos\phi$ for $\Delta\lambda$.

2. Special Test Data

No. of photos:	6
Strips:	2
No. of ground points:	1
Method of solution:	Jordan diagonalization
No. of iterations:	1
Weights:	all one

Results

	$\Delta\phi$	$\Delta\lambda$	Δh
Gt. Pt.	-0.001	-0.001	0.094
Ex. St. 1	-0.001	0.000	0.002
2	-0.001	0.000	0.002
3	-0.001	0.000	0.001
4	-0.001	0.000	0.003
5	-0.001	0.000	0.003
6	-0.001	0.000	0.002

The discrepancies $\Delta\phi$ $\Delta\lambda$ Δh in the above results are not beyond machine truncation errors associated with similar errors in the input data, nor are they beyond the convergence limit reached in the last iteration. A comparison of the elements of the orientation matrices of the exposure stations as computed by the program and as computed by an independent program which generated the fictitious data 2 showed an agreement of 4 digits on the average. This corresponds to a discrepancey in the angular orientation of the camera station

of the order of magnitude of 10^{-4} radians. The solution for the incremental angular displacements, dT, of the camera stations are of the order of magnitude of 10^{-5} radians. This corresponds to a discrepancy on the ground of 0.2 ft. (the scale of data 2 is $\frac{1}{20,000}$) and is in agreement with the $\Delta\phi$, $\Delta\lambda$, and Δh obtained. Note, however, that the solution is computed after the solution for the dT's is used to correct the camera orientations so that the final discrepancy can be of the order of 0.2 ft. or much smaller and in fact beyond the capacity of the computer. The values of the incremental displacements dR's are of the order of magnitude of 10^{-3} ft. and clearly do not affect the accuracy of the solution. We can therefore conclude that, for the fictitious data tests run by this laboratory, the results are satisfactory and the small discrepancies between input control data and output is in the range of the truncation errors and convergence limits.

B. REAL DATA

1. Test 10

a. No. of photos: 15
No. of strips: 3
No. of ground pts: 64

Method of solution: Jordan diagonalization
No. of iterations: 4
Weights: all one

Results:

Gr. Pt.	Type	$\Delta\phi$	$\Delta\lambda$	Δh
9204	A	-0.025	-0.004	4.326
9107	A	0.010	-0.020	-3.145
9403	A	0.055	0.040	-1.438
9407	A	0.011	-0.039	0.532
9603	A	-0.018	-0.019	-2.133
9606	A	-0.026	0.058	1.982

b. No. of photos: 8
No. of strips: 2
No. of ground pts: 37

Method of solution: Jordan diagonalization
No. of iterations: 4
Weights: all one

Results:

Gr. Pt.	Type	$\Delta\phi$	$\Delta\lambda$	Δh
9403	A	0.034	0.031	0.074
9603	A	-0.034	-0.003	-0.020
9606	A	-0.012	0.030	0.080
9407	A	appears on one photo only		

c. No. of photos: 8
No. of strips: 2
No. of ground points: 37

Method of solution: Stiefel
No. of iterations: 6
No. of cycles per iteration: 3 and 2

Results: for 3 cycles per iteration

Gr. Pt.	Type	$\Delta\phi$	$\Delta\lambda$	Δh
9403	A	0.024	0.067	5.806
9603	A	-0.005	0.037	-7.788
9606	A	0.020	-0.004	2.239

Results: for 2 cycles per iteration

9403	A	-0.011	0.131	6.117
9603	A	0.023	0.100	-5.353
9606	A	0.060	-0.053	-0.997

2. Test 19 HALCON

No. of photos: 24
No. of strips: 3
No. of ground points: 140

a. Method of solution: Jordan diagonalization
No. of iterations: 5
Weights. all one

Results:

Gr. Pt.	Type	$\Delta\phi$	$\Delta\lambda$	Δh
302	A	0.005	0.316	28.535
340	A	0.224	-0.070	-20.862
223	A	-0.055	0.060	- 7.405
104	A	-0.066	-0.194	-27.321
152	A	-0.109	-0.155	27.119
all points type B		0.003	0.316	28.651
		0.223	-0.072	-20.463
		-0.056	0.061	- 7.169
		-0.066	-0.196	-27.052
		-0.110	-0.153	27.516

b. Method of solution: Jordan diagonalization

No. of iterations: 6 and no convergence weights

Unique points coordinates: of pass points: 1; of control points: 8

Ground control coordinates: >10000

Input = output of 2.a

Results:

Gr. Pt.	Type	$\Delta\phi$	$\Delta\lambda$	Δh
302	A	-0.009	0.005	0.286
340	A	0.015	-0.002	-0.561
223	A	0.013	0.006	5.341
104	A	-0.033	-0.004	0.346
152	A	0.008	-0.023	1.279

The solution for dT's and dR's showed that after the second iteration they alternate in sign without change in order of magnitude:

maximum dT $\sim$ 0.01 radian
maximum dR $\sim$ 1000 ft.

The results of 1 indicate that the Stiefel method of solution compares very poorly with the Jordan method. We will disregard it in the following as it introduces significant errors foreign to the photogrammetric problem.

The results in 1.a. indicate that the discrepancies $\Delta\phi$ and $\Delta\lambda$ when

transferred into equivalent feet are generally of the same order of magnitude as Δh with the maximum being approximately 5 ft. These results must be considered as satisfactory considering that the systematic errors have not been corrected and that information is not available on the probable accuracy of the input data. The orders of magnitude of the dT's and dR's of the last iteration are 10^{-5} radians and 1 ft. respectively, and the scale of the model is $\frac{1}{20,000}$.

The results of 1.b. indicates smaller $\Delta\phi$'s, $\Delta\lambda$'s, and Δh's for the same ground control as 1.a. They show further that the Δh's are significantly smaller than the $\Delta\phi$'s and $\Delta\lambda$'s in equivalent feet. This merely reflects the facts that the data in 1.b. is only slightly redundant as regard to altitude information. The lower degree of control redundancy is A.b. as compared to A.a. would also be the reason for smaller $\Delta\phi$'s and $\Delta\lambda$'s in 1.b.

In 2.a. the discrepancies reach 30 ft. When one considers that with strong control, such as in 2.b., the $\Delta\phi$'s, $\Delta\lambda$'s, and Δh's become very small in comparison with the results of 2.a., even without convergence, it becomes apparent that the data contains significant errors. A.a. shows that with a uniform weighting it is impossible to fit the ground control without a "stretching" in the horizontal direction reaching 30 ft., and 1.b. shows that if one enforces strongly the ground control along with the corresponding image coordinates, the adjustment of the remaining image coordinates in each iteration is so high as to prevent convergence. The errors in the data are of two kinds: ground control errors and image coordinates errors. As was pointed out previously,

this laboratory has no knowledge of their order of magnitude or of the occurrence of these errors. Something can be said, however, about atmospheric refraction errors. For an altitude of 30000 ft., an angle with the vertical of 45° (which takes account of tilt) and a focal length of 1 ft., the atmospheric refraction error on a vertical photograph is about 50 microns which in turn produces an apparent ground displacement of about 5 ft. Other systematic errors, such as film shrinkage and lens distortion, increase the probable error. This accounts for at least part of the values of the $\Delta\phi$'s, $\Delta\lambda$'s, and Δh's. The remaining part would be caused by errors in the ground control coordinates. As the least squares method is not intended to adjust for large systematic errors, the results of 2.a. are not conclusive. The results of 2.b., however, show that the weighting does influence significantly the solution and can even prevent convergence when it is not in accordance with the actual accuracy of the data. The Δh's are seen to be in general of the same order of magnitude as the $\Delta\phi$'s and $\Delta\lambda$'s expressed in feet. This would likely hold also for pass points in which case the general method of least squares, independently of an accurate weighting, would present an improvement on previous photogrammetric solutions.

X. RECOMMENDATIONS

The possibilities and flexibility of the least squares method should be studied further. In order to do this, it is recommended that a routine to compute the residuals or adjustment to the input data and the sum of the weighted squares of these residuals be included in the program. For data corrected for systematic errors and whose random errors do noton the average exceed a certain limit, the additional computer time and storage area required may overshadow the resulting gain in probable accuracy.

However, for gaining a knowledge about the order of magnitude of the errors contained in a set of data when not available otherwise as well as for obtaining a more accurate solution when important errors cannot be corrected beforehand, the calculation of the residuals would be the only alternative.

Consider the following. A set of data with minimum ground control but redundant as regards the relative orientation of the camera stations is run. At each iteration the adjustment of the data as given by the least squares method required to make the rays intersect as well as the sum of the squares of the residuals and their standard deviation can be computed. An iteration is reached where the above sum and the corresponding standard deviation are minimum.

For truly unrelated random errors, the standard deviation computed as outlined above is a true standard deviation in the probability theory sense. For errors of any kind throughout the entire data set, an estimate

is provided of the order of magnitude of the errors affecting the image points coordinates. A concentration of errors in parts of the data can be discovered by isolating different combinations of redundant subsets of the data and comparing the corresponding standard deviations. When minimum ground control is used, no discrepancy, beyond truncation errors and convergence limits, between input and output values of the ground control coordinates will occur, but when obtaining the ground coordinates of the redundant ground control points which were used only as pass points, discrepancies will occur. If they cannot be accounted for by a reasonable error in the images coordinates, they provide an estimate of the accuracy of the ground control information.

Although not recommended as sound practice but still useful when no other course is possible, uncorrected data can be adjusted using the computed residuals given by the last iteration before the solution is computed. Finally, when systematic errors are removed from the data and information on the standard deviations of the random errors is available, the adjusted input data can be used to obtain high accuracy results.

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